6.013, Electromagnetics and Applications
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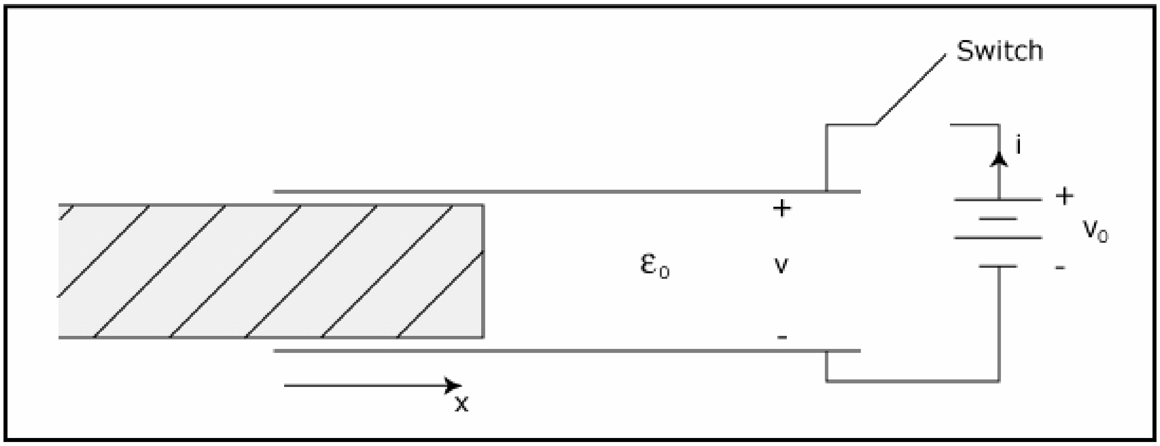
September 27 and 29, 2005
Lectures 6 and 7: Polarization, Conduction, and Magnetization

I. Experimental Observation: Dielectric Media

A. Fixed Voltage - Switch Closed $(v = V_o)$



As an insulating material enters a free-space capacitor at constant voltage more charge flows onto the electrodes; i.e., as x increases, i increases.

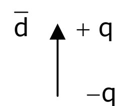
B. Fixed Charge - Switch open (i=0)

As an insulating material enters a free space capacitor at constant charge, the voltage decreases; i.e., as x increases, v decreases.

II. Dipole Model of Polarization

A. Polarization Vector $\overline{P} = N\overline{p} = Nq\overline{d}$ $\quad(\overline{p} = q\overline{d}$ dipole moment)

N dipoles/Volume ($\overline{P}$ is dipole density)



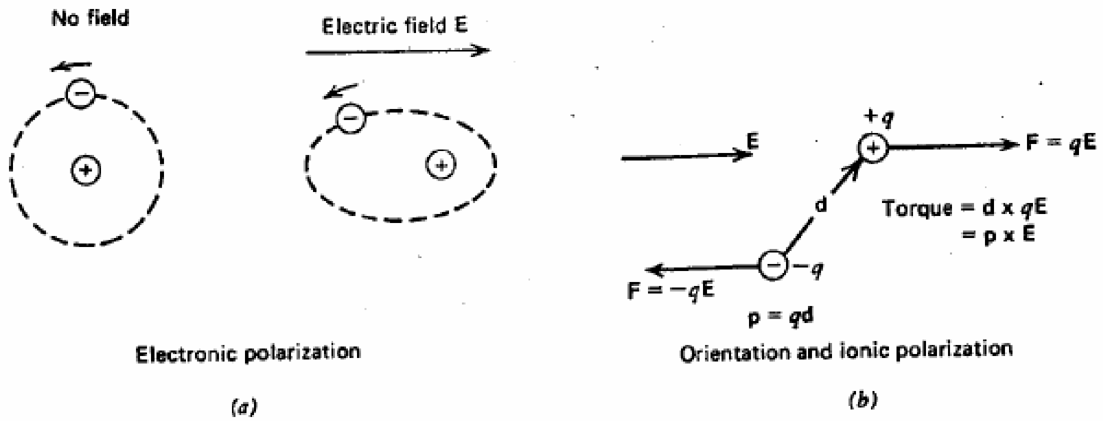
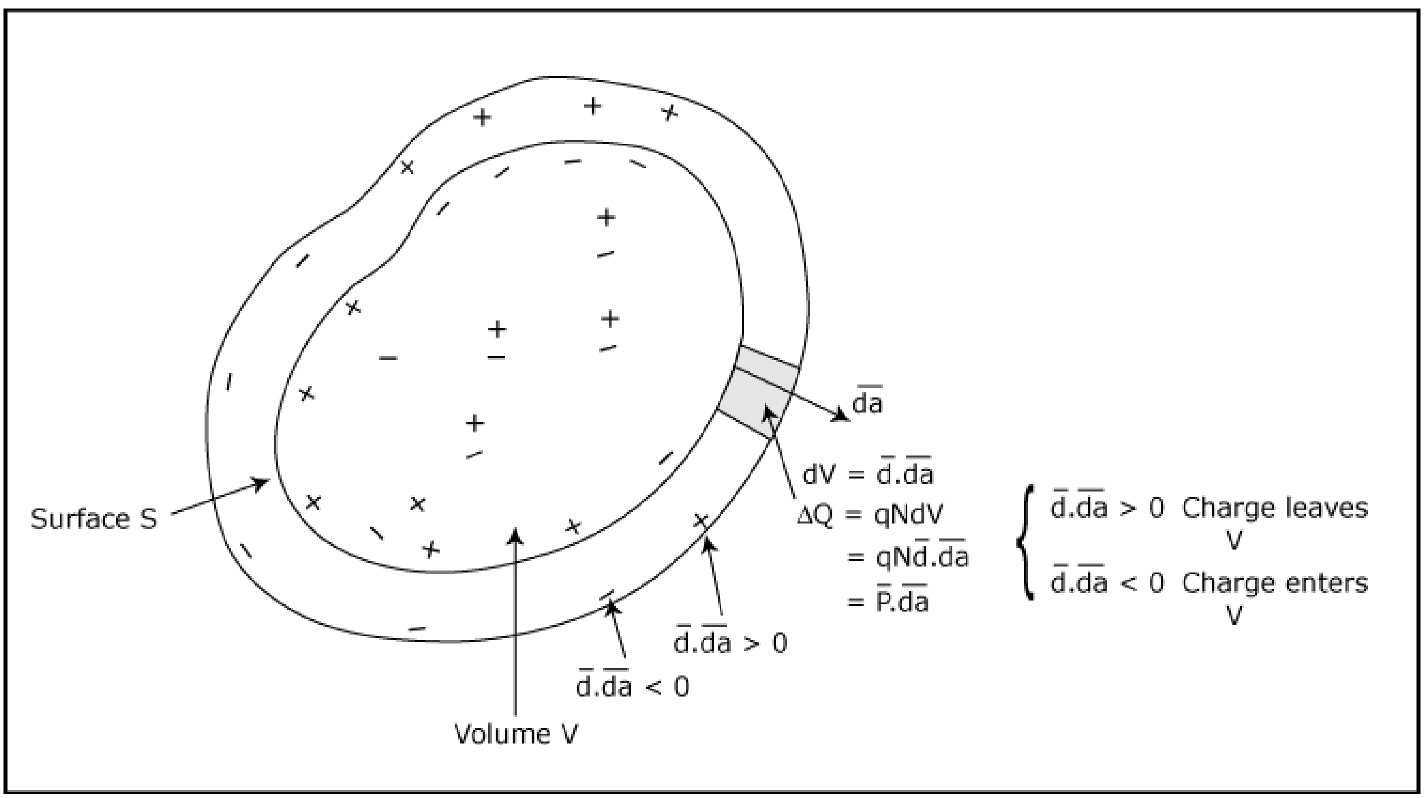


Figure 3-1 An electric dipole consists of two charges of equal magnitude but opposite sign, separated by a small vector distance **d**. (*a*) Electronic polarization arises when the average motion of the electron cloud about its nucleus is slightly displaced. (*b*) Orientation polarization arises when an asymmetric polar molecule tends to line up with an applied electric field. If the spacing **d** also changes, the molecule has ionic polarization.



$$Q_{\text{inside V}} = -\oint_S qN\overline{d}\bullet\overline{da} = \underbrace{\int_V \rho_p \, dV}$$

paired charge or
equivalently
polarization
charge density

$$Q_{\text{inside V}} = -\oint_S \overline{P}\bullet\overline{da} = -\int_V \nabla\bullet\overline{P}\, dV = \int_V \rho_P \, dV \qquad \text{(Divergence Theorem)}$$

$$\overline{P} = qN\overline{d} = N\overline{p}$$

$$\nabla\bullet\overline{P} = -\rho_P$$

B. Gauss' Law

$$\nabla\bullet\left(\varepsilon_o \overline{E}\right) = \rho_{\text{total}} = \rho_u + \rho_P = \rho_u - \nabla\bullet\overline{P}$$

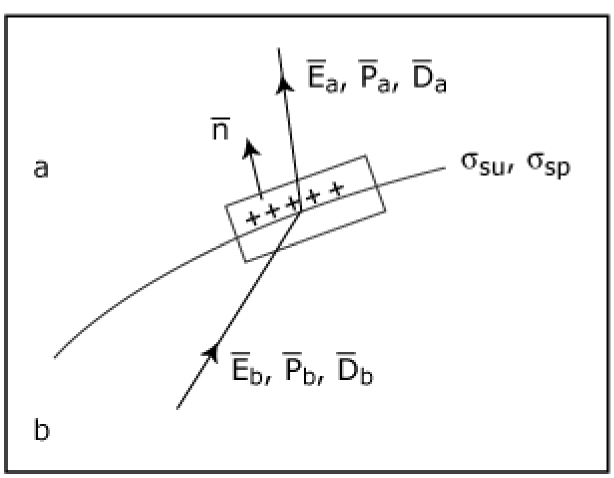
Unpaired charge
density; also
called free charge
density

$$\nabla\bullet\left(\varepsilon_o \overline{E} + \overline{P}\right) = \rho_u$$

$$\overline{D} = \varepsilon_o \overline{E} + \overline{P} \qquad \text{Displacement Flux Density}$$

$$\nabla\bullet\overline{D} = \rho_u$$

C. Boundary Conditions



$$\nabla \cdot \overline{D} = \rho_u \Rightarrow \oint_S \overline{D} \cdot \overline{da} = \int_V \rho_u \, dV \Rightarrow \overline{n} \cdot \left[\overline{D}_a - \overline{D}_b\right] = \sigma_{su}$$

$$\nabla \cdot \overline{P} = -\rho_P \Rightarrow \oint_S \overline{P} \cdot \overline{da} = -\int_V \rho_P \, dV \Rightarrow \overline{n} \cdot \left[\overline{P}_a - \overline{P}_b\right] = -\sigma_{sp}$$

$$\nabla \cdot \left(\varepsilon_o \overline{E}\right) = \rho_u + \rho_P \Rightarrow \oint_S \varepsilon_o \overline{E} \cdot \overline{da} = \int_V \left(\rho_u + \rho_P\right) dV \Rightarrow \overline{n} \cdot \varepsilon_o \left[\overline{E}_a - \overline{E}_b\right] = \sigma_{su} + \sigma_{sp}$$

D. Polarization Current Density

$$\Delta Q = q N \, dV = q N \, \overline{d} \cdot \overline{da} = \overline{P} \cdot \overline{da}$$

[Amount of Charge passing through surface area element $\overline{da}$]

$$d i_p = \frac{\partial \Delta Q}{\partial t} = \frac{\partial \overline{P}}{\partial t} \cdot \overline{da}$$

[Current passing through surface area element $\overline{da}$]

$$= \overline{J}_p \cdot \overline{da}$$

polarization current density

$$\overline{J}_p = \frac{\partial \overline{P}}{\partial t}$$

Ampere's law:

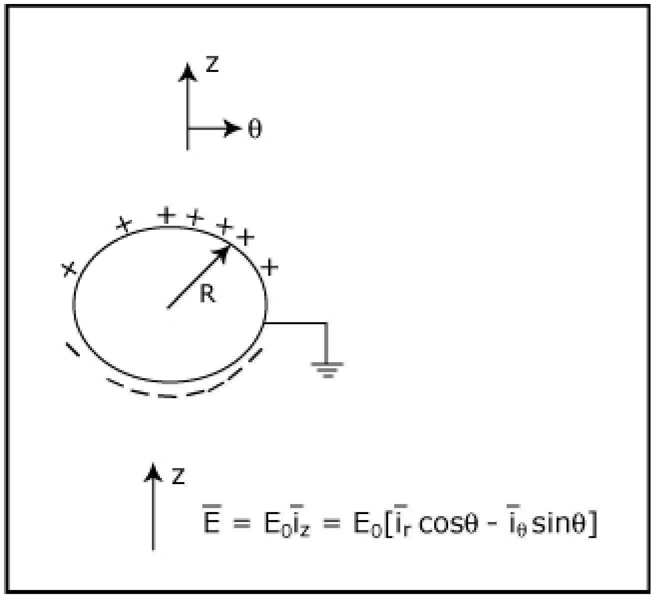
$$\nabla \times \overline{H} = \overline{J}_u + \overline{J}_p + \varepsilon_o \frac{\partial \overline{E}}{\partial t}$$

$$= \overline{J}_u + \frac{\partial \overline{P}}{\partial t} + \varepsilon_o \frac{\partial \overline{E}}{\partial t}$$

$$= \overline{J}_u + \frac{\partial}{\partial t}\left(\varepsilon_o \overline{E} + \overline{P}\right)$$

$$= \overline{J}_u + \frac{\partial \overline{D}}{\partial t}; \quad \overline{D} = \varepsilon_0 \overline{E} + \overline{P}$$

III. Equipotential Sphere in a Uniform Electric Field



$$\lim_{r\to\infty} \Phi(r,\theta) = -E_o r\cos\theta \qquad \left[\Phi = -E_o z = -E_o r\cos\theta\right]$$

$$\Phi(r = R, \theta) = 0$$

$$\Phi(r,\theta) = -E_o\left[r - \frac{R^3}{r^2}\right]\cos\theta$$

This solution is composed of the superposition of a uniform electric field plus the field due to a point electric dipole at the center of the sphere:

$$\Phi_{\text{dipole}} = \frac{p\ \cos\theta}{4\pi\varepsilon_o r^2} \qquad \text{with } p = 4\pi\varepsilon_o E_o R^3$$

This dipole is due to the surface charge distribution on the sphere.

$$\sigma_s(r = R,\ \theta) = \varepsilon_o E_r(r = R,\ \theta) = -\varepsilon_o \left.\frac{\partial\Phi}{\partial r}\right|_{r=R} = \varepsilon_o E_o\left[1 + \left.\frac{2R^3}{r^3}\right|_{r=R}\right]\cos\theta$$

$$= 3\varepsilon_o E_o \cos\theta$$

IV. Artificial Dielectric

$$E = \frac{v}{d}, \quad \sigma_s = \varepsilon E = \frac{\varepsilon v}{d}$$

$$q = \sigma_s A = \frac{\varepsilon A}{d} v$$

$$C = \frac{q}{v} = \frac{\varepsilon A}{d}$$

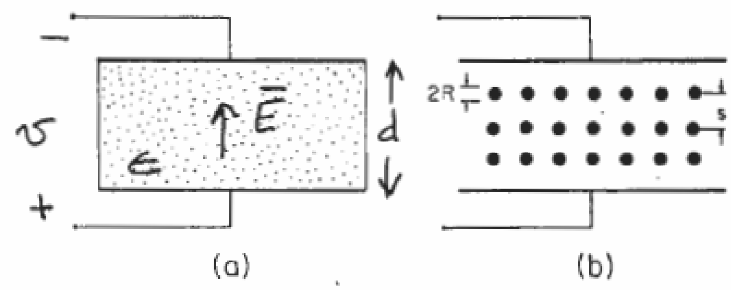


Figure 6.6.1 (a) Plane parallel capacitor with region between electrodes occupied by a dielectric. (b) Artificial dielectric composed of cubic array of perfectly conducting spheres having radius R and spacing s.

For spherical array of non-interacting spheres (s >> R)

$$\bar{P} = 4\pi\varepsilon_o R^3 E_o \bar{i}_z \Rightarrow P_z = N p_z = 4\pi\varepsilon_o R^3 E_o N$$

$$N = 1/s^3$$

$$\bar{P} = \varepsilon_o \underbrace{\left[4\pi\left(\frac{R}{s}\right)^3\right]}_{\psi_e \text{ (electric susceptibility)}} \bar{E} = \psi_e \varepsilon_o \bar{E} \qquad \left(\psi_e = 4\pi\left(\frac{R}{s}\right)^3\right)$$

$$\bar{D} = \varepsilon_o \bar{E} + \bar{P} = \varepsilon_o \underbrace{\left[1 + \psi_e\right]}_{\varepsilon_r \text{ (relative dielectric constant)}} \bar{E} = \varepsilon \bar{E}$$

$$\varepsilon = \varepsilon_r \varepsilon_o = \varepsilon_o\left[1 + \psi_e\right] = \varepsilon_o\left(1 + 4\pi\left(\frac{R}{s}\right)^3\right)$$

V. Demonstration: Artificial Dielectric

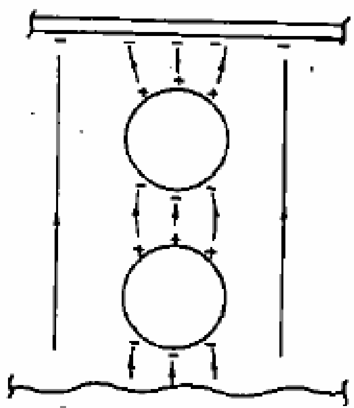


Figure 6.6.2 From the microscopic point of view, the increase in capacitance results because the dipoles adjacent to the electrode induce image charges on the electrode in addition to those from the unpaired charges on the opposite electrode.

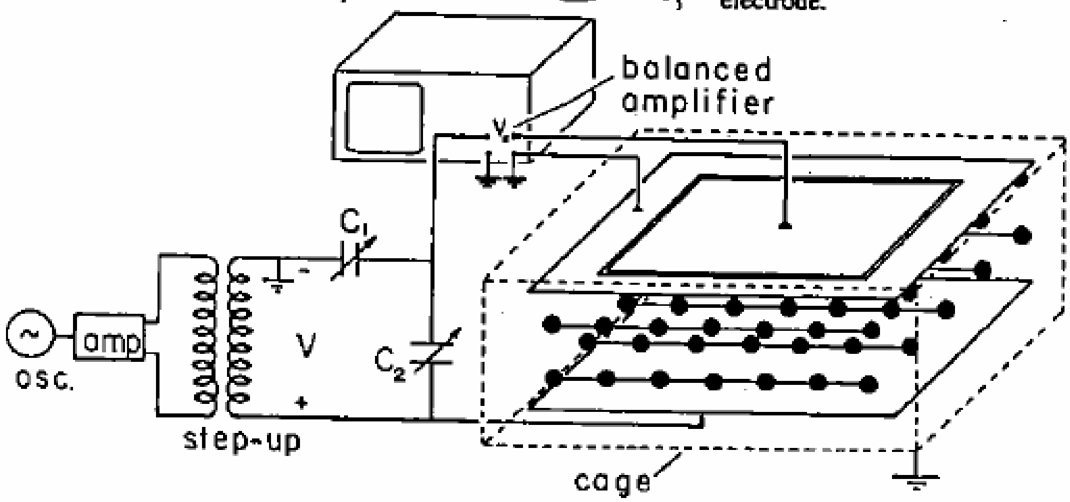
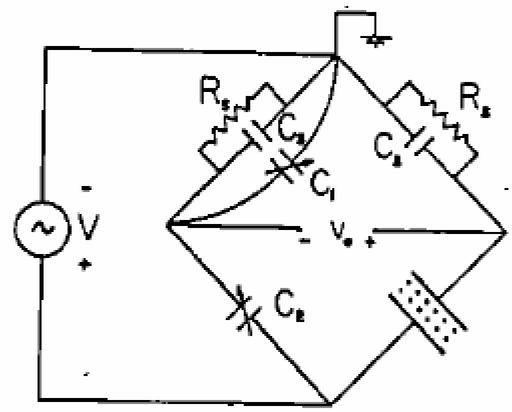


Figure 6.6.3 Demonstration in which change in capacitance is used to measure the equivalent dielectric constant of an artificial dielectric.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.



$$v_o = R_s \omega (\Delta C) V$$

Figure 6.6.4 Balanced amplifiers of oscilloscope, balancing capacitors, and demonstration capacitor shown in Figure 6.6.4 comprise the elements in the bridge circuit. The driving voltage comes from the transformer, while v_o is the oscilloscope voltage.

$$E = \frac{v}{d} \Rightarrow \sigma_s = \varepsilon E = \frac{\varepsilon v}{d}$$

$$q = \sigma_s A = \frac{\varepsilon A}{d} v \Rightarrow C = \frac{q}{v} = \frac{\varepsilon A}{d}$$

$$|\Delta i| = \omega \Delta C \quad |V| = \frac{v_o}{R_s}$$

$$\Delta C = \frac{(\varepsilon - \varepsilon_o) A}{d} = 4 \pi \varepsilon_o \left(\frac{R}{s}\right)^3 \frac{A}{d}$$

$R=1.87$ cm, $s=8$ cm, $A= (0.4)^2$ m^2, $d=0.15$m

$\omega = 2\pi(250$ Hz$)$, $R_s=100$ kΩ, $V=566$ volts peak

$\Delta C=1.5$ pf

$$v_0 = \omega \Delta C R_s |V|$$

$$=(2\pi)\ (250)\ (1.5 \times 10^{-12})\ (10^5)\ 566 = 0.135 \text{ volts}$$

VI. Plasma Conduction Model (Classical)

$$m_+ \frac{d\bar{v}_+}{dt} = q_+ \bar{E} - m_+ \nu_+ \bar{v}_+ - \frac{\nabla p_+}{n_+}$$

$$m_- \frac{d\bar{v}_-}{dt} = -q_- \bar{E} - m_- \nu_- \bar{v}_- - \frac{\nabla p_-}{n_-}$$

$$p_+ = n_+ kT \ , \ \ p_- = n_- kT$$

$k=1.38 \times 10^{-23}$ joules/^{o}K Boltzmann Constant

A. London Model of Superconductivity $[T \to 0, \nu_\pm \to 0]$

$$m_+ \frac{d\bar{v}_+}{dt} = q_+ \bar{E} \quad , \quad m_- \frac{d\bar{v}_-}{dt} = -q_- \bar{E}$$

$$\bar{J}_+ = q_+ n_+ \bar{v}_+ \qquad , \qquad \bar{J}_- = -q_- n_- \bar{v}_-$$

$$\frac{d\bar{J}_+}{dt} = \frac{d}{dt}\left(q_+ n_+ \bar{v}_+\right) = q_+ n_+ \frac{d\bar{v}_+}{dt} = q_+ n_+ \frac{\left(q_+ \bar{E}\right)}{m_+} = \underbrace{\frac{q_+^{\,2} n_+}{m_+}}_{\omega_{p+}^2 \varepsilon} \bar{E}$$

$$\frac{d\bar{J}_-}{dt} = -\frac{d}{dt}\left(q_- n_- \bar{v}_-\right) = -q_- n_- \frac{d\bar{v}_-}{dt} = -q_- n_- \frac{\left(-q_- \bar{E}\right)}{m_-} = \underbrace{\frac{q_-^{\,2} n_-}{m_-}}_{\omega_{p-}^2 \varepsilon} \bar{E}$$

$$\omega_{p+}^2 = \frac{q_+^{\,2} n_+}{m_+ \varepsilon} \quad , \quad \omega_{p-}^2 = \frac{q_-^{\,2} n_-}{m_- \varepsilon} \qquad \text{(} \omega_p = \text{plasma frequency)}$$

For electrons: $q_-=1.6 \times 10^{-19}$ Coulombs, $m_-=9.1 \times 10^{-31}$ kg

$n_-=10^{20}/m^3$, $\varepsilon = \varepsilon_o \approx 8.854 \times 10^{-12}$ farads/m

$$\omega_{p-} = \sqrt{\frac{q_-^{\,2} n_-}{m_- \varepsilon}} \approx 5.6 \times 10^{11} \text{ rad/s}$$

$$f_{p-} = \frac{\omega_{p-}}{2\pi} \approx 9 \times 10^{10} \text{ Hz}$$

B. Drift-Diffusion Conduction [Neglect inertia]

$$m_+ \overbrace{\cancel{\frac{d\bar{v}_+}{dt}}}^{0} = q_+ \bar{E} - m_+ \nu_+ \bar{v}_+ - \frac{\nabla(n_+ kT)}{n_+} \Rightarrow \bar{v}_+ = \frac{q_+}{m_+ \nu_+} \bar{E} - \frac{kT}{m_+ \nu_+ n_+} \nabla n_+$$

$$m_- \overbrace{\cancel{\frac{d\bar{v}_-}{dt}}}^{0} = -q_- \bar{E} - m_- \nu_- \bar{v}_- - \frac{\nabla(n_- kT)}{n_-} \Rightarrow \bar{v}_- = \frac{-q_-}{m_- \nu_-} \bar{E} - \frac{kT}{m_- \nu_- n_-} \nabla n_-$$

$$\bar{J}_+ = q_+ n_+ \overline{v_+} = \frac{q_+^2 n_+}{m_+ \nu_+} \bar{E} - \frac{q_+ kT}{m_+ \nu_+} \nabla n_+$$

$$\bar{J}_- = -q_- n_- \overline{v_-} = \frac{q_-^2 n_-}{m_- \nu_-} \bar{E} + \frac{q_- kT}{m_- \nu_-} \nabla n_-$$

$$\rho_+ = q_+ n_+ \ , \quad \rho_- = -q_- n_-$$

$$\bar{J}_+ = \rho_+ \mu_+ \bar{E} - D_+ \nabla \rho_+$$

$$\bar{J}_- = -\rho_- \mu_- \bar{E} - D_- \nabla \rho_-$$

$$\mu_+ = \frac{q_+}{m_+ \nu_+} \quad , \quad D_+ = \frac{kT}{m_+ \nu_+}$$

$$\underbrace{\mu_- = \frac{q_-}{m_- \nu_-}}_{\text{charge molulities}} \quad , \quad \underbrace{D_- = \frac{kT}{m_- \nu_-}}_{\text{Molecular Diffusion Coefficients}}$$

$$\underbrace{\frac{D_+}{\mu_+} = \frac{D_-}{\mu_-} = \frac{kT}{q}}_{\text{Einstein's Relation}} = \text{thermal voltage } (25 \text{ mV@ } T \approx 300^{o} \text{K})$$

C. Drift-Diffusion Conduction Equilibrium $\left(\bar{J}_+ = \bar{J}_- = 0\right)$

$$\bar{J}_+ = 0 = \rho_+ \mu_+ \bar{E} - D_+ \nabla\rho_+ = -\rho_+ \mu_+ \nabla\Phi - D_+ \nabla\rho_+$$

$$\bar{J}_- = 0 = -\rho_- \mu_- \bar{E} - D_- \nabla\rho_- = \rho_- \mu_- \nabla\Phi - D_- \nabla\rho_-$$

$$\nabla\Phi = -\frac{D_+}{\rho_+ \mu_+}\nabla\rho_+ = \frac{-kT}{q}\nabla\left(\ln\rho_+\right)$$

$$\nabla\Phi = \frac{D_-}{\rho_- \mu_-}\nabla\rho_- = \frac{kT}{q}\nabla\left(\ln\rho_-\right)$$

$$\left.\begin{array}{l} \rho_+ = \rho_o e^{-q\Phi/kT} \\ \rho_- = -\rho_o e^{+q\Phi/kT} \end{array}\right\} \quad \text{Boltzmann Distributions}$$

$\rho_+\left(\Phi = 0\right) = -\rho_-\left(\Phi = 0\right) = \rho_o$ [Potential is zero when system is charge neutral]

$$\nabla^2\Phi = \frac{-\rho}{\varepsilon} = -\frac{\left(\rho_+ + \rho_-\right)}{\varepsilon} = \frac{-\rho_o}{\varepsilon}\left[e^{-q\Phi/kT} - e^{+q\Phi/kT}\right] = \frac{2\rho_o}{\varepsilon}\sinh\frac{q\Phi}{kT}$$

(Poisson - Boltzmann Equation)

Small Potential Approximation: $\frac{q\Phi}{kT} << 1$

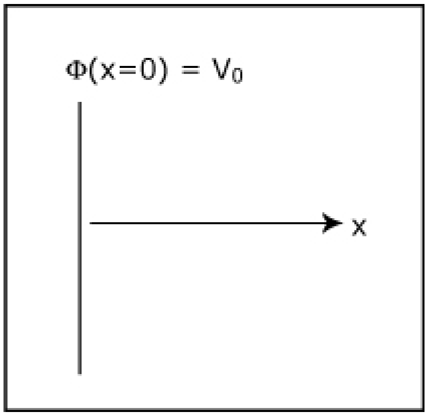
$$\sinh\frac{q\Phi}{kT} \approx \frac{q\Phi}{kT}$$

$$\nabla^2\Phi - \frac{2\rho_0 q}{\varepsilon kT}\Phi = 0$$

$$\nabla^2\Phi - \frac{\Phi}{L_d^2} = 0 \ ; \quad L_d = \sqrt{\frac{\varepsilon kT}{2\rho_o q}} \qquad \text{Debye Length}$$

D. Case Studies

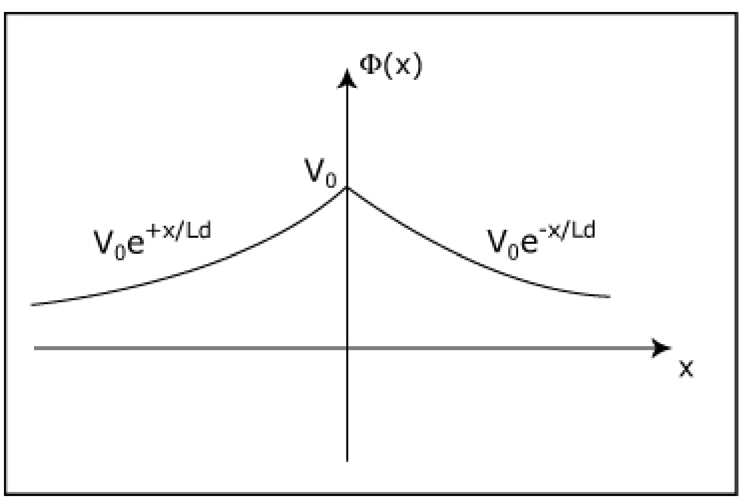
1. Planar Sheet



$$\frac{d^2\Phi}{dx^2} - \frac{\Phi}{L_d^{\,2}} = 0 \quad \Rightarrow \quad \Phi = A_1\, e^{x/L_d} + A_2\, e^{-x/L_d}$$

$$\text{B.C.} \quad \Phi(x \to \pm\infty) = 0 \qquad \Rightarrow \Phi(x) = \begin{cases} V_o\, e^{-x/L_d} & x > 0 \\ V_o\, e^{+x/L_d} & x < 0 \end{cases}$$

$$\Phi(x = 0) = V_o$$



$$E_x = -\frac{d\Phi}{dx} = \begin{cases} \dfrac{V_o}{L_d} e^{-x/L_d} & x > 0 \\[2ex] -\dfrac{V_o}{L_d} e^{x/L_d} & x < 0 \end{cases}$$

$$\rho = \varepsilon \frac{dE_x}{dx} = \begin{cases} -\dfrac{\varepsilon V_o}{L_d^{\,2}} e^{-x/L_d} & x > 0 \\[2ex] -\dfrac{\varepsilon V_o}{L_d^{\,2}} e^{+x/L_d} & x < 0 \end{cases}$$

$$\sigma_s (x = 0) = \varepsilon \left[E_x (x = 0_+) - E_x (x = 0_-) \right] = \frac{2 \varepsilon V_o}{L_d}$$

2. Point Charge (Debye Shielding)

$$\underbrace{\nabla^2 \Phi}_{} - \frac{\Phi}{L_d^{\,2}} = 0 \qquad \Rightarrow \qquad \frac{d^2}{dr^2}(r\Phi) - \frac{r\Phi}{L_d^2} = 0 \quad 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) \qquad r\Phi = A_1 e^{-r/Ld} + \cancel{A_2} e^{+r/L_d}$$

$$\Phi(r) = \frac{Q}{4\pi\varepsilon r} e^{-r/L_d}$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)$$

E. Ohmic Conduction

$$\bar{J}_+ = \rho_+ \mu_+ \bar{E} - D_+ \nabla \rho_+$$

$$\bar{J}_- = -\rho_- \mu_- \bar{E} - D_- \nabla \rho_-$$

If charge density gradients small, then $\nabla \rho_\pm$ negligible $\Rightarrow \rho_+ = -\rho_- = \rho_o$

$$\bar{J} = \bar{J}_+ + \bar{J}_- = (\rho_+ \mu_+ - \rho_- \mu_-) E = \underbrace{\rho_o (\mu_+ + \mu_-)}_{\sigma \,=\, \text{ohmic conductivity}} \bar{E} = \sigma \bar{E}$$

$\bar{J} = \sigma \bar{E}$ (Ohm's Law)

F. p—n Junction Diode

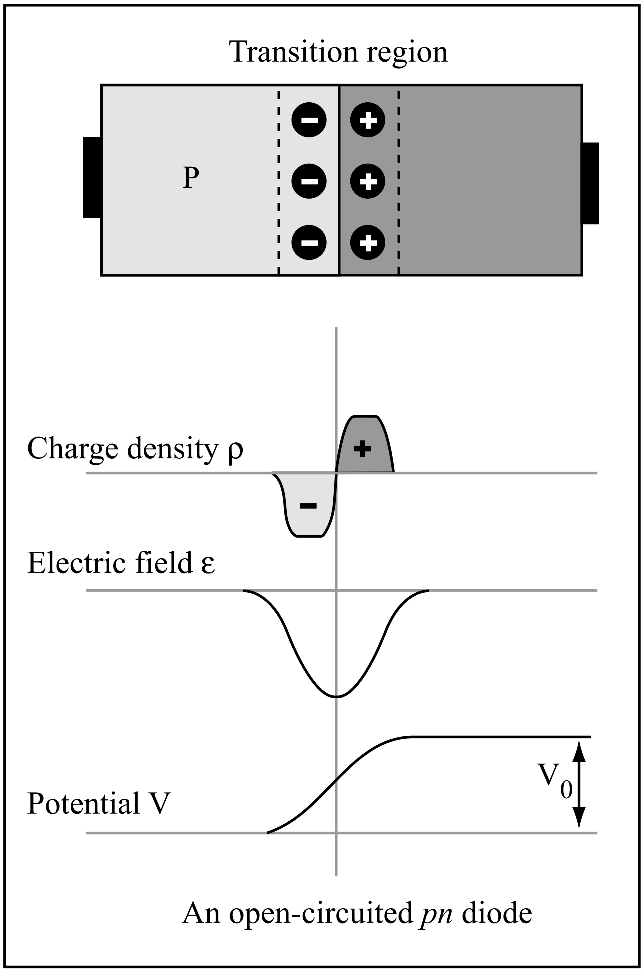


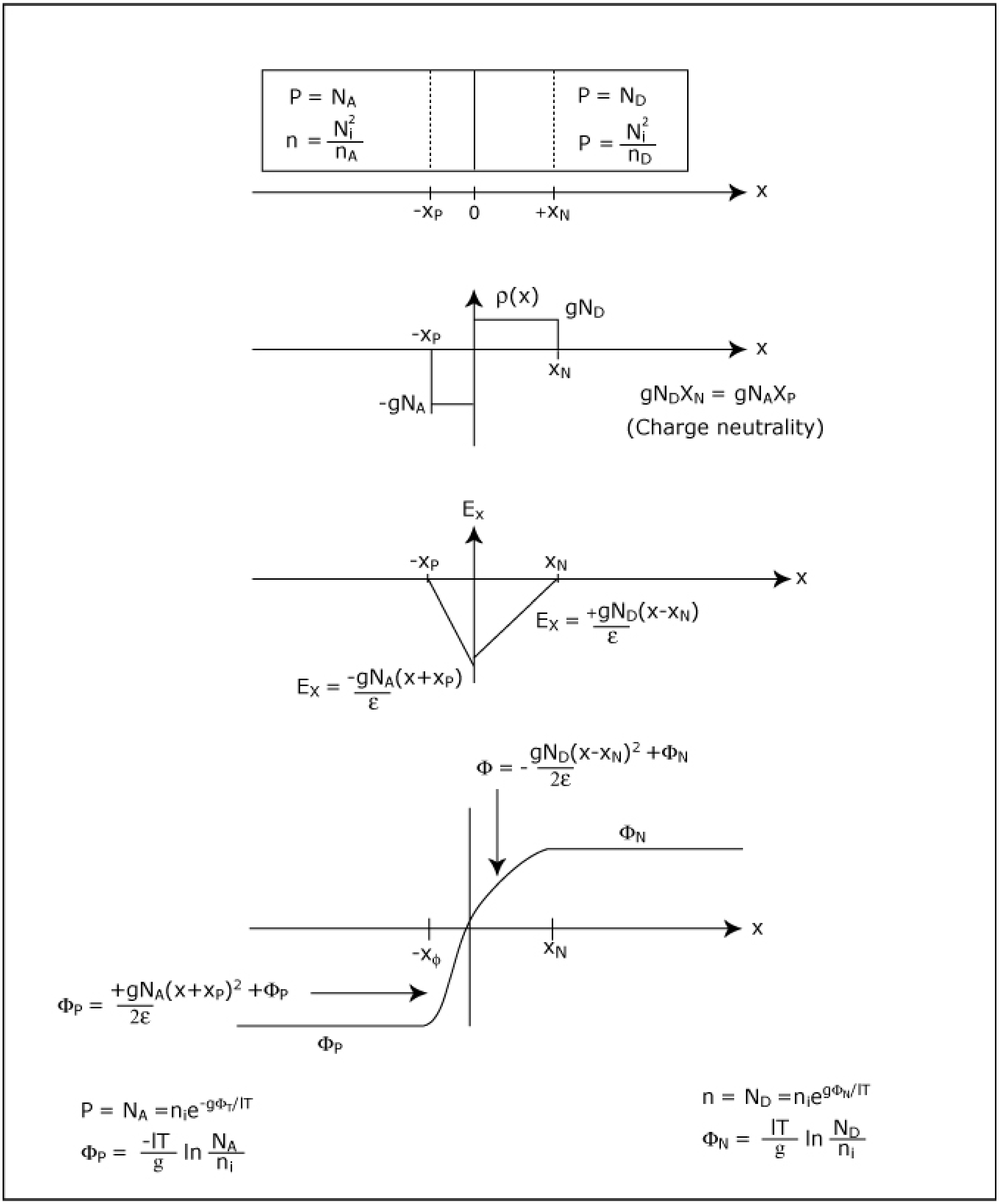
Figure by MIT OpenCourseWare.

$$\Delta\Phi = \Phi_n - \Phi_p = \frac{kT}{q}\ln\frac{N_A N_D}{n_i^2}$$

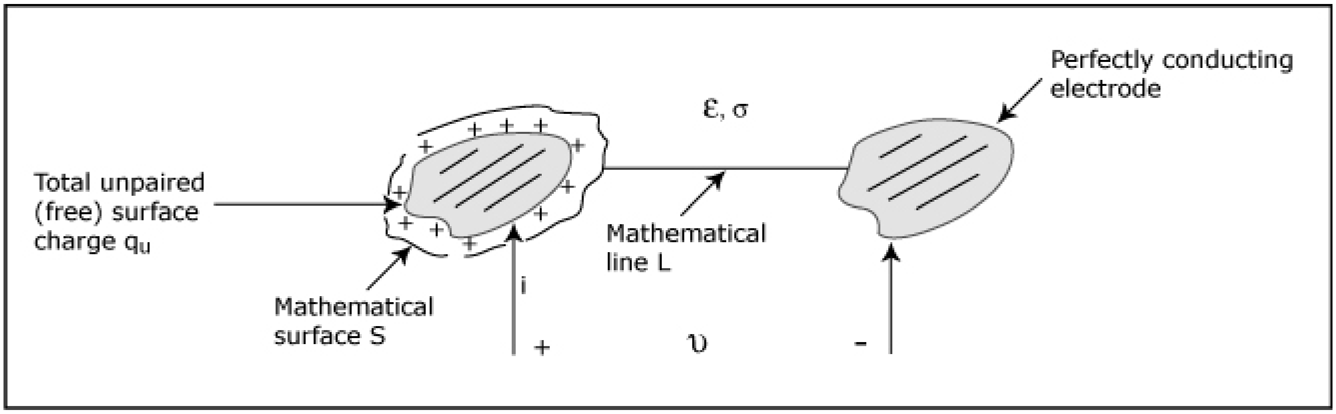
$$\Phi(X=0) = \Phi_p + \frac{qN_A x_p^2}{2\varepsilon} = \Phi_n - \frac{qN_D x_n^2}{2\varepsilon}$$

$$\Delta\Phi = \Phi_n - \Phi_p = \frac{qN_D x_n^2}{2\varepsilon} + \frac{qN_A x_p^2}{2\varepsilon}$$

$$= \frac{qN_D x_n}{2\varepsilon}\left(x_n + x_p\right)$$


P = N_A
n = N_i^2 / n_A
P = N_D
P = N_i^2 / n_D
x
-x_P
0
+x_N
ρ(x)
gN_D
-x_P
x_N
-gN_A
gN_D X_N = gN_A X_P
(Charge neutrality)
E_x
-x_P
x_N
E_X = +gN_D(x-x_N)/ε
E_X = -gN_A(x+x_P)/ε
Φ = - gN_D(x-x_N)^2/2ε + Φ_N
Φ_N
-x_ϕ
x_N
Φ_P = +gN_A(x+x_P)^2/2ε + Φ_P
Φ_P
P = N_A = n_i e^{-gΦ_T/lT}
Φ_P = -lT/g ln N_A/n_i
n = N_D = n_i e^{gΦ_N/lT}
Φ_N = lT/g ln N_D/n_i

VII. Relationship Between Resistance and Capacitance In Uniform Media Described by ε and σ.



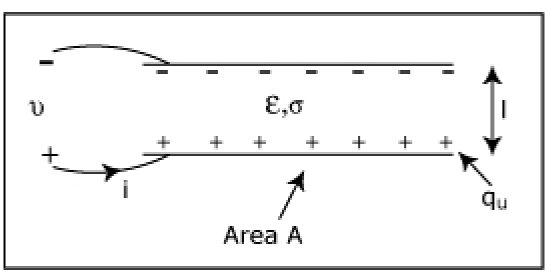
$$C = \frac{q_u}{v} = \frac{\oint_S \overline{D} \cdot \overline{da}}{\int_L \overline{E} \cdot \overline{ds}} = \frac{\varepsilon \oint_S \overline{E} \cdot \overline{da}}{\int_L \overline{E} \cdot \overline{ds}}$$

$$R = \frac{v}{i} = \frac{\int_L \overline{E} \cdot \overline{ds}}{\oint_S \overline{J} \cdot \overline{da}} = \frac{\int_L \overline{E} \cdot \overline{ds}}{\sigma \oint_S \overline{E} \cdot \overline{da}}$$

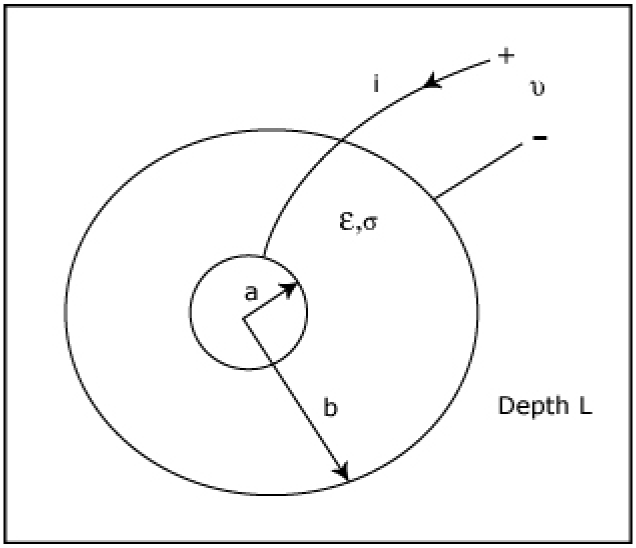
$$RC = \frac{\int_L \overline{E} \cdot \overline{ds}}{\sigma \oint_S \overline{E} \cdot \overline{da}} \quad \frac{\varepsilon \oint_L \overline{E} \cdot \overline{da}}{\oint_L \overline{E} \cdot \overline{ds}} = \frac{\varepsilon}{\sigma}$$

Check:

Parallel Plate Electrodes: $R = \frac{l}{\sigma A}$, $C = \frac{\varepsilon A}{l} \Rightarrow RC = \varepsilon/\sigma$

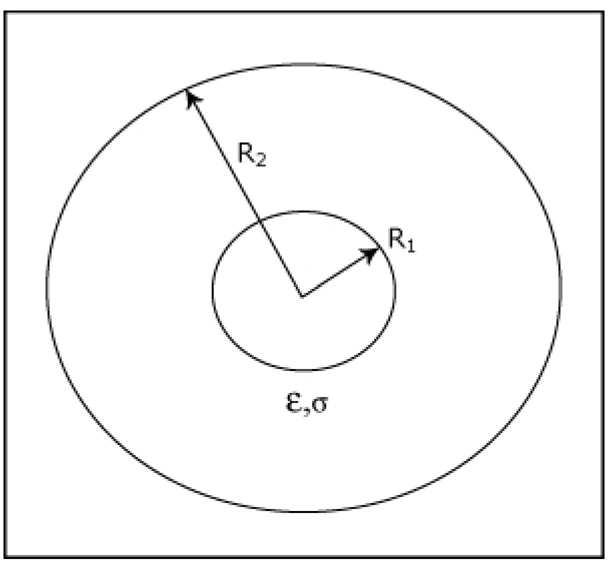


Coaxial



$$R = \frac{\ln b/a}{2\pi\sigma l}, \quad C = \frac{2\pi\varepsilon l}{\ln b/a} \Rightarrow RC = \varepsilon/\sigma$$

Concentric Spherical



$$R = \frac{\frac{1}{R_1} - \frac{1}{R_2}}{4\pi\sigma}, \quad C = \frac{4\pi\varepsilon}{\frac{1}{R_1} - \frac{1}{R_2}} \Rightarrow RC = \varepsilon/\sigma$$

VIII. Charge Relaxation in Uniform Conductors

$$\nabla \cdot \bar{J}_u + \frac{\partial \rho_u}{\partial t} = 0$$

$$\nabla \cdot \bar{E} = \rho_u / \varepsilon$$

$$\bar{J}_u = \sigma \bar{E}$$

$$\sigma \underbrace{\nabla \cdot \bar{E}}_{\rho_u/\varepsilon} + \frac{\partial \rho_u}{\partial t} = 0 \Rightarrow \frac{\partial \rho_u}{\partial t} + \frac{\sigma}{\varepsilon} \rho_u = 0$$

$$\tau_e = \varepsilon/\sigma \text{ = dielectric relaxation time}$$

$$\frac{\partial \rho_u}{\partial t} + \frac{\rho_u}{\tau_e} = 0 \Rightarrow \rho_u = \rho_0 \left(r,\, t=0\right) e^{-t/\tau_e}$$

IX. Demonstration 7.7.1 – Relaxation of Charge on Particle in Ohmic Conductor

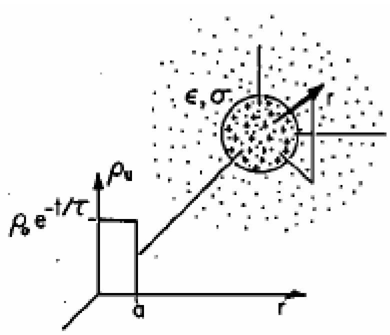


Figure 7.7.1 Within a material having uniform conductivity and permittivity, initially there is a uniform charge density ρ_u in a spherical region, having radius a. In the surrounding region the charge density is given to be initially zero and found to be always zero. Within the spherical region, the charge density is found to decay exponentially while retaining its uniform distribution.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

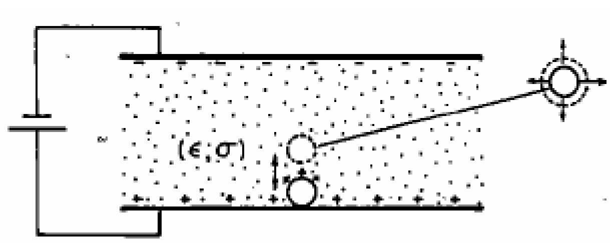
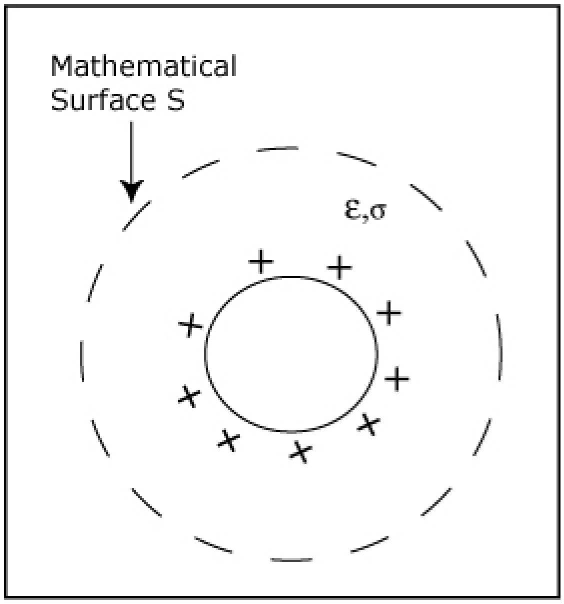


Figure 7.7.2 The region between plane parallel electrodes is filled by a semi-insulating liquid. With the application of a constant potential difference, a metal particle resting on the lower plate makes upward excursions into the fluid. [See footnote 1.]

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.



$$\oint_S \bar{J}\cdot\overline{da} = \sigma\oint_S \bar{E}\cdot\overline{da} = \frac{\sigma q_u}{\varepsilon} = \frac{-dq}{dt}$$

$$\frac{dq}{dt}+\frac{q}{\tau_e}=0 \Rightarrow q = q(t=0)\,e^{-t/\tau_e} \qquad \left(\tau_e = \varepsilon/\sigma\right)$$

Partially Uniformly Charged Sphere

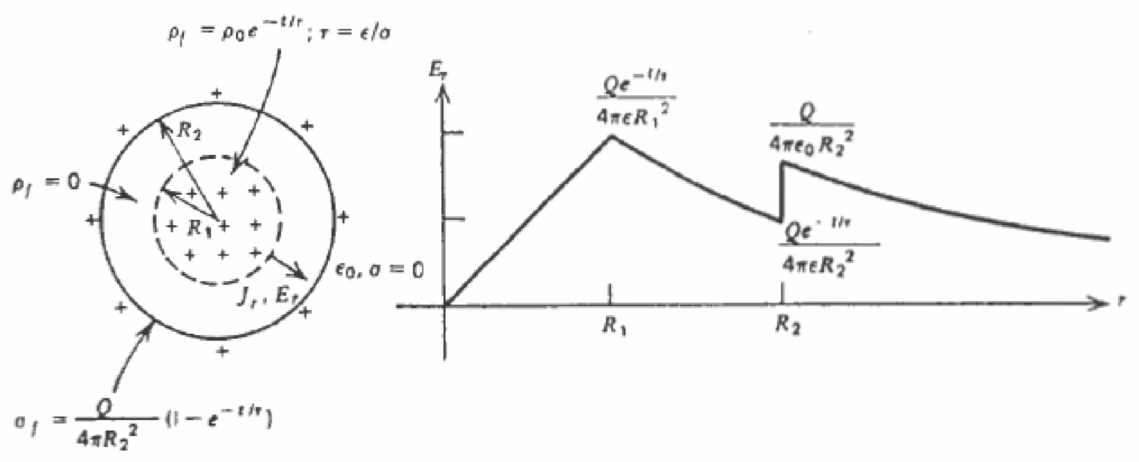


Figure 3-21 An initial volume charge distribution within an Ohmic conductor decays exponentially towards zero with relaxation time $\tau = \epsilon/\sigma$ and appears as a surface charge at an interface of discontinuity. Initially uncharged regions are always uncharged with the charge transported through by the current.

$$\rho_u(t=0) = \begin{cases} \rho_0 & r < R_1 \\ 0 & r > R_1 \end{cases} \qquad Q_T = \frac{4}{3}\pi R_1^{\,3}\rho_0$$

$$\rho_u(t) = \begin{cases} \rho_0\, e^{-t/\tau_e} & r < R_1 \\ 0 & r > R_1 \end{cases} \qquad (\tau_e = \varepsilon/\sigma)$$

$$E_r(r,t) = \begin{cases} \dfrac{\rho_0\, r\, e^{-t/\tau_e}}{3\varepsilon} = \dfrac{Q\, r\, e^{-t/\tau_e}}{4\pi\varepsilon R_1^{\,3}} & 0 < r < R_1 \\[2ex] \dfrac{Q\, e^{-t/\tau_e}}{4\pi\varepsilon r^2} & R_1 < r < R_2 \\[2ex] \dfrac{Q}{4\pi\varepsilon_0 r^2} & r > R_2 \end{cases}$$

$$\sigma_{su}(r=R_2) = \varepsilon_0 E_r(r=R_{2+}) - \varepsilon\ E_r(r=R_{2-})$$

$$= \frac{Q}{4\pi R_2^{\,2}}\left(1 - e^{-t/\tau_e}\right)$$

X. Self-Excited Water Dynamos

A. DC High Voltage Generation (Self-Excited)

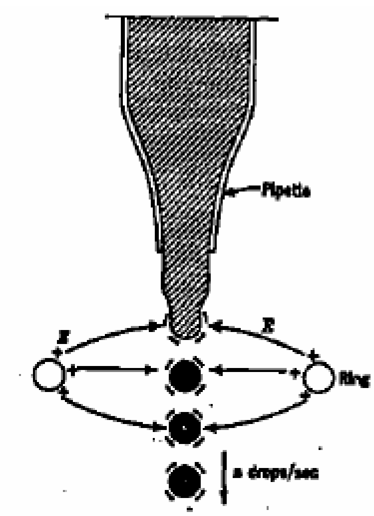
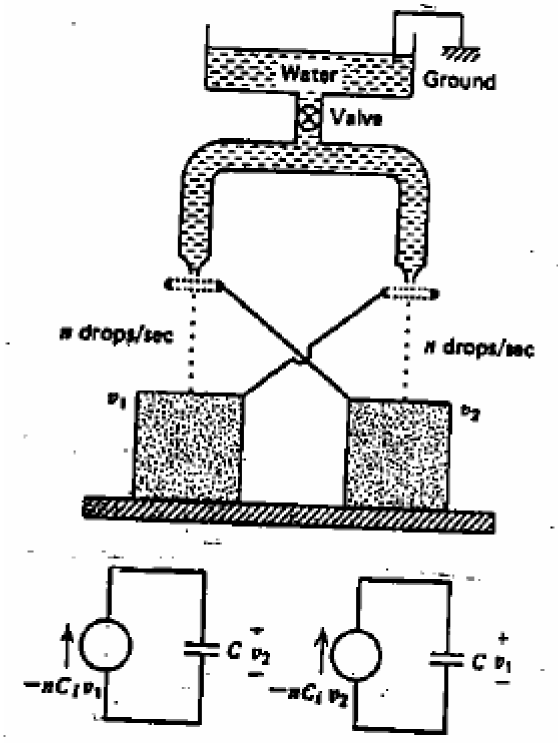


Fig. 7.2.12 As the drops form near the rings in Fig. 7.2.11, image charges are induced on the surface of the water. When the drops are completely formed (and insulated from one another) they carry a net charge, the sign of which is determined by the sign of the charge on the respective rings.

Courtesy of Herbert Woodson and James Melcher. Used with permission.
Woodson, Herbert H., and James R. Melcher. *Electromechanical Dynamics, Part 2: Fields, Forces, and Motion*. Malabar, FL: Krieger Publishing Company, 1968. ISBN: 9780894644597.



From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

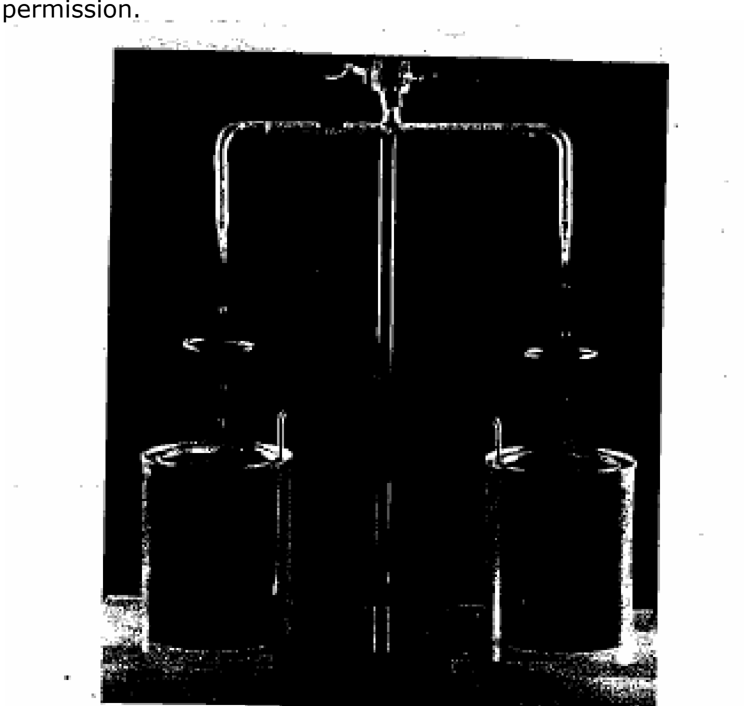


Fig. 7.2.11a Water drops fall into the cans through cross-connected wire loops. A potential difference of more than 20 kV between cans is spontaneously generated by the motion of the drops. For optimum operation the drops should form nearer to the rings than shown. This is accomplished by increasing the flow rate.

Courtesy of Herbert Woodson and James Melcher. Used with permission.
Woodson, Herbert H., and James R. Melcher. *Electromechanical Dynamics, Part 2: Fields, Forces, and Motion*. Malabar, FL: Krieger Publishing Company, 1968. ISBN: 9780894644597.

$$-nC_i v_1 = C\frac{dv_2}{dt} \qquad v_1 = \hat{V}_1 e^{st} \qquad -nC_i\hat{V}_1 = Cs\hat{V}_2$$
$$\Rightarrow \qquad\qquad \Rightarrow$$
$$-nC_i v_2 = C\frac{dv_1}{dt} \qquad v_2 = \hat{V}_2 e^{st} \qquad -nC_i\hat{V}_2 = Cs\hat{V}_1$$

$$\underbrace{\begin{bmatrix} \frac{nC_i}{Cs} & 1 \\ 1 & \frac{nC_i}{Cs} \end{bmatrix}}_{\text{Det} = 0}\begin{bmatrix}\hat{V}_1 \\ \hat{V}_2\end{bmatrix} = 0$$

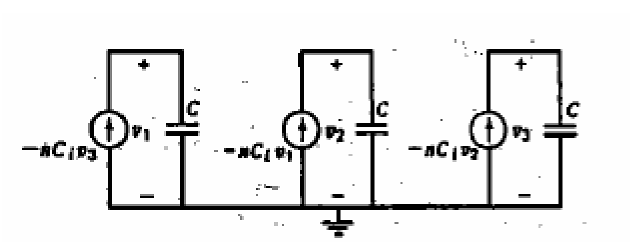
$$\left(\frac{nC_i}{Cs}\right)^2 = 1 \Rightarrow s = \pm\frac{nC_i}{C}$$

$\oplus$ root blows up

$e^{\frac{nC_i}{C}t}$

Any perturbation grows exponentially with time

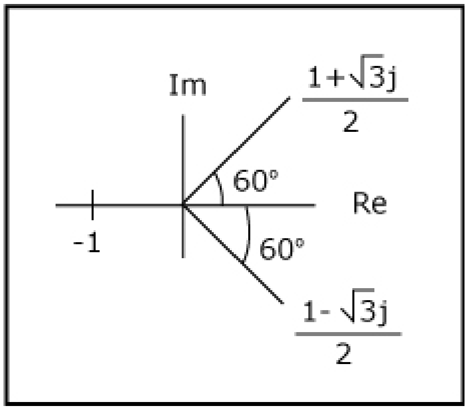
B. AC High Voltage Self – Excited Generation



From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$-nC_i v_1 = C\frac{dv_2}{dt} \quad ; \quad v_1 = \hat{V}_1 e^{st}$$
$$-nC_i v_2 = C\frac{dv_3}{dt} \qquad v_2 = \hat{V}_2 e^{st}$$
$$-nC_i v_3 = C\frac{dv_1}{dt} \qquad v_3 = \hat{V}_3 e^{st}$$

$$\underbrace{\begin{bmatrix} nC_i & Cs & 0 \\ 0 & nC_i & Cs \\ Cs & 0 & nC_i \end{bmatrix}}_{\det = 0}\begin{bmatrix}\hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3\end{bmatrix} = 0$$



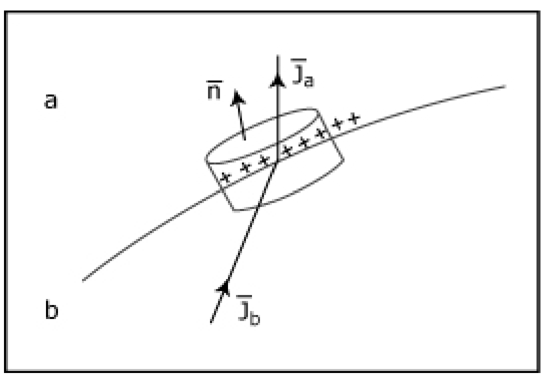
$$(nC_i)^3 + (Cs)^3 = 0 \Rightarrow s = \left(\frac{nC_i}{C}\right)(-1)^{1/3}$$

$s_1 = -nC_i/C$ (exponentially decaying solution)

$$(-1)^{1/3} = -1, \frac{1 \pm \sqrt{3}j}{2}$$

$s_{2,3} = \frac{nC_i}{2C}\left[1 \pm \sqrt{3}\,j\right]$ (blows up exponentially because $s_{real} > 0$; but also oscillates at frequency $s_{imag} \neq 0$)

XI. Conservation of Charge Boundary Condition

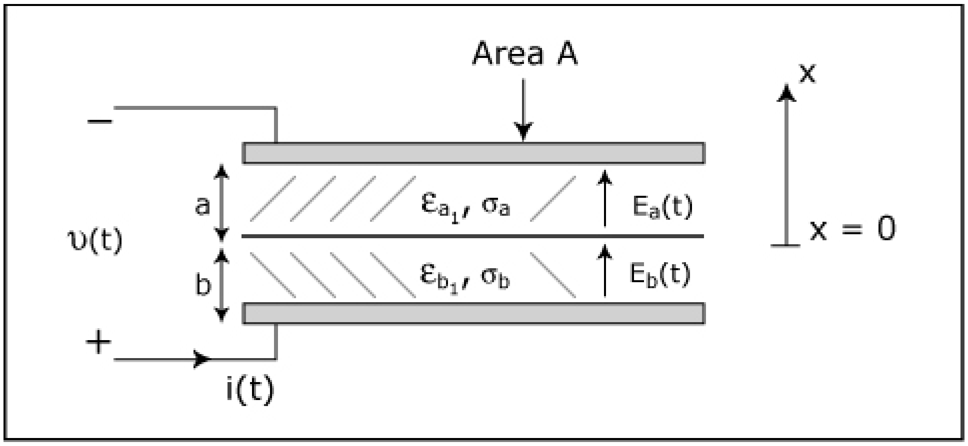


$$\nabla \cdot \bar{J}_u + \frac{\partial \rho_u}{\partial t} = 0$$

$$\oint_S \bar{J}_u \cdot \overline{da} + \frac{d}{dt}\int_V \rho_u \, dV = 0$$

$$\bar{n} \cdot \left[\bar{J}_a - \bar{J}_b\right] + \frac{d}{dt}\sigma_{su} = 0$$

XII. Maxwell Capacitor



A. General Equations

$$\bar{E} = \begin{cases} E_a(t)\,\bar{i}_x & 0 < x < a \\ E_b(t)\,\bar{i}_x & -b < x < 0 \end{cases}$$

$$\int_{-b}^{a} E_x\, d_x = v(t) = E_b(t)b + E_a(t)a$$

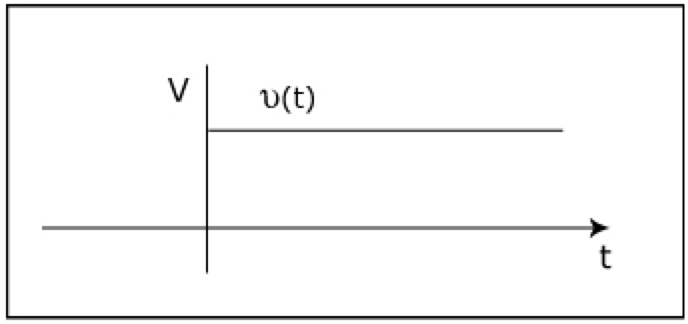
$$\bar{n}\bullet\left[\bar{J}_a - \bar{J}_b\right] + \frac{d\sigma_{su}}{dt} = 0 \Rightarrow \sigma_a E_a(t) - \sigma_b E_b(t) + \frac{d}{dt}\left[\varepsilon_a E_a(t) - \varepsilon_b E_b(t)\right] = 0$$

$$E_b(t) = \frac{v(t)}{b} - E_a(t)\frac{a}{b}$$

$$\sigma_a E_a(t) - \sigma_b\left[\frac{v(t)}{b} - E_a(t)a\right] + \frac{d}{dt}\left[\varepsilon_a E_a(t) - \varepsilon_b\left(\frac{v(t)}{b} - E_a(t)a\right)\right] = 0$$

$$\left(\varepsilon_a + \frac{\varepsilon_b a}{b}\right)\frac{dE_a}{dt} + \left(\sigma_a + \frac{\sigma_b a}{b}\right)E_a(t) = \frac{\sigma_b\, v(t)}{b} + \frac{\varepsilon_b}{b}\,\frac{dv}{dt}$$

B. Step Voltage: $v(t) = V\ u(t)$



Then $\dfrac{dv}{dt} = V\ \delta(t)$ (an impulse)

At t=0

$$\left(\varepsilon_a + \frac{\varepsilon_b a}{b}\right)\frac{dE_a}{dt} = \frac{\varepsilon_b}{b}\frac{dv}{dt} = \frac{\varepsilon_b}{b}\ V\delta(t)$$

Integrate from $t=0_-$ to $t=0_+$

$$\int_{t=0_-}^{t=0_+}\left(\varepsilon_a + \frac{\varepsilon_b a}{b}\right)\frac{dE_a}{dt}\,dt = \left(\varepsilon_a + \frac{\varepsilon_b a}{b}\right)E_a\Big|_{t=0_-}^{t=0_+} = \int_{t=0_-}^{0_+}\frac{\varepsilon_b}{b}\ V\delta(t)\,dt = \frac{\varepsilon_b}{b}\ V$$

$$E_a(t = 0_-) = 0$$

$$\left(\varepsilon_a + \frac{\varepsilon_b a}{b}\right)E_a(t = 0_+) = \frac{\varepsilon_b}{b}V \Rightarrow E_a(t = 0_+) = \frac{\varepsilon_b\ V}{\varepsilon_b\, b + \varepsilon_b\, a}$$

For $t > 0$, $\dfrac{dv}{dt} = 0$

$$\left(\varepsilon_a + \frac{\varepsilon_b a}{b}\right)\frac{dE_a}{dt} + \left(\sigma_a + \frac{\sigma_b a}{b}\right)E_a(t) = \frac{\sigma_b}{b}V$$

$$E_a(t) = \frac{\sigma_b\ V}{\sigma_a b + \sigma_b a} + A\,e^{-t/\tau} \quad ; \quad \tau = \frac{\varepsilon_a b + \varepsilon_b a}{\sigma_a b + \sigma_b a}$$

$$E_a(t = 0) = \frac{\sigma_b\ V}{\sigma_a b + \sigma_b a} + A\ = \frac{\varepsilon_b\ V}{\varepsilon_a b + \varepsilon_b a} \Rightarrow A\ = V\left[\frac{\varepsilon_b}{\varepsilon_a b + \varepsilon_b a} - \frac{\sigma_b}{\sigma_a b + \sigma_b a}\right]$$

$$E_a(t) = \frac{\sigma_b\ V}{\sigma_a b + \sigma_b a}\left(1 - e^{-t/\tau}\right) + \frac{\varepsilon_b\ V}{\varepsilon_a b + \varepsilon_b a}\,e^{-t/\tau}$$

$$E_b(t) = \frac{V}{b} - E_a(t)\frac{a}{b}$$

$$\sigma_{su}(t) = \varepsilon_a E_a(t) - \varepsilon_b E_b(t) = \varepsilon_a E_a(t) - \varepsilon_b \left(\frac{V}{b} - \frac{a}{b} E_a(t)\right)$$

$$= E_a(t)\left(\varepsilon_a + \frac{\varepsilon_b a}{b}\right) - \varepsilon_b \frac{V}{b}$$

$$= \frac{V(\sigma_b \varepsilon_a - \sigma_a \varepsilon_b)}{(\sigma_a b + \sigma_b a)}\left(1 - e^{-t/\tau}\right)$$

C. Sinusoidal Steady State: $v(t) = \mathrm{Re}\left[\hat{V} e^{j\omega t}\right]$

$$E_a(t) = \mathrm{Re}\left[\widehat{E_a} e^{j\omega t}\right]$$

$$E_b(t) = \mathrm{Re}\left[\widehat{E_b} e^{j\omega t}\right]$$

Conservation of Charge Interfacial Boundary Condition

$$\sigma_a E_a(t) - \sigma_b E_b(t) + \frac{d}{dt}\left[\varepsilon_a E_a(t) - \varepsilon_b E_b(t)\right] = 0$$

$$\widehat{E_a}\left[\sigma_a + j\omega\varepsilon_a\right] - \widehat{E_b}\left[\sigma_b + j\omega\varepsilon_b\right] = 0$$

$$\widehat{E_b}\, b + \widehat{E_a}\, a = \hat{V}$$

$$\widehat{E_b} = \frac{\hat{V}}{b} - \frac{\widehat{E_a}\, a}{b}$$

$$\widehat{E_a} = \left[\sigma_a + j\omega\varepsilon_a\right] - \left(\frac{\hat{V}}{b} - \frac{\widehat{E_a}\, a}{b}\right)\left[\sigma_b + j\omega\varepsilon_b\right] = 0$$

$$\widehat{E_a} = \left[\sigma_a + j\omega\varepsilon_a + \frac{a}{b}\left(\sigma_b + j\omega\varepsilon_b\right)\right] = \frac{\hat{V}}{b}\left[\sigma_b + j\omega\varepsilon_b\right] = 0$$

$$\frac{\widehat{E_a}}{j\omega\varepsilon_b + \sigma_b} = \frac{\widehat{E_b}}{j\omega\varepsilon_a + \sigma_a} = \frac{\hat{V}}{\left[b\left(\sigma_a + j\omega\varepsilon_a\right) + a\left(\sigma_b + j\omega\varepsilon_b\right)\right]}$$

$$\hat{\sigma}_{su} = \varepsilon_a \widehat{E_a} - \varepsilon_b \widehat{E_b}$$

$$= \frac{\left(\varepsilon_a\sigma_b - \varepsilon_b\sigma_a\right)}{\left[b\left(\sigma_a + j\omega\varepsilon_a\right) + a\left(\sigma_b + j\omega\varepsilon_b\right)\right]}\hat{V}$$

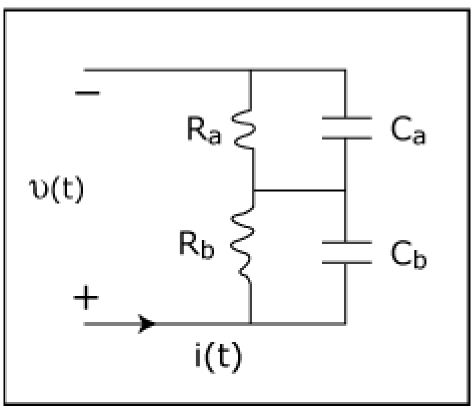
D. Equivalent Circuit (Electrode Area A)

$$\hat{I} = \left(\sigma_a + j\omega\varepsilon_a\right)\widehat{E_a}\,A = \left(\sigma_b + j\omega\varepsilon_b\right)\widehat{E_b}\,A$$

$$= \frac{\hat{V}}{\dfrac{R_a}{R_a C_a j\omega + 1} + \dfrac{R_b}{R_b C_b j\omega + 1}}$$

$$R_a = \frac{a}{\sigma_a A}\,, \quad R_b = \frac{b}{\sigma_b A}$$

$$C_a = \frac{\varepsilon_a A}{a}\,, \quad C_b = \frac{\varepsilon_b A}{b}$$



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XIII. Magnetic Dipoles

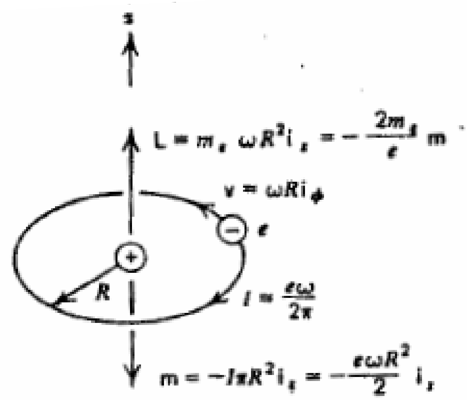

Figure 5-16 The orbiting electron has its magnetic moment **m** in the direction opposite to its angular momentum **L** because the current is opposite to the electron's velocity.

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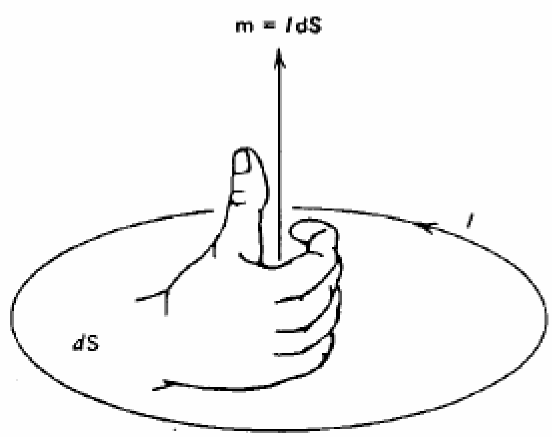

Figure 5-14 A magnetic dipole consists of a small circulating current loop. The magnetic moment is in the direction normal to the loop by the right-hand rule.

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Diamagnetism

$$I = \frac{e}{2\pi/\omega} = \frac{e\,\omega}{2\pi}\qquad , \quad \overline{m} = -I\,\pi R^2\,\bar{i}_z = \frac{-e\,\omega}{2\cancel{\pi}}\cancel{\pi} R^2\,\bar{i}_z = \frac{-e\,\omega R^2}{2}\,\bar{i}_z$$

Angular Momentum $\overline{L} = m_e R\,\bar{i}_r \times \overline{v} = m_e R\,(\omega R)\left(\bar{i}_r \times \bar{i}_\phi\right) = m_e\,\omega R^2\,\bar{i}_z$

$$\left(\bar{r}\times\bar{p}\right) \qquad\qquad = -\frac{2m_e}{e}\overline{m}$$

linear momentum

L is quantized in units of $\frac{h}{2\pi}$, $h = 6.62\times 10^{-34}$ joule – sec

(Planck's constant)

$$\left|\overline{m}\right| = \frac{e\left|\overline{L}\right|}{2\,m_e} = \frac{e\,h}{2\,\pi(2)m_e} = \underbrace{\frac{e h}{4\,\pi\,m_e} \approx 9.3\times 10^{-24}}_{\text{Bohr magneton } m_B} \text{ amp} - m^2$$

Bohr magneton m_B
(smallest unit of
magnetic moment)

Imagine all Bohr magnetons in sphere of radius R aligned. Net magnetic moment is

$$m = m_B \underbrace{\left(\frac{4}{3}\pi R^3 \rho\right)}_{\substack{\text{Total mass}\\ \text{of sphere}}} \frac{A_0}{M_0}$$

A_0 — Avogadro's number = 6.023 x 10^{26} molecules per kilogram–mole

M_0 — molecular weight

For iron: ρ =7.86 x 10^3 kg/m^3, M_0=56

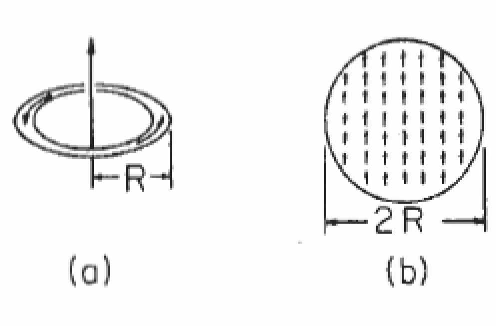


Figure 9.0.1 (a) Current i in loop of radius R gives dipole moment **m**. (b) Spherical material of radius R has dipole moment approximated as the sum of atomic dipole moments.

For a current loop

$$m = i\pi R^2 = m_B \frac{4}{3}\pi R^3 \rho \frac{A_0}{M_0} \Rightarrow i = m_B \frac{4}{3} R \rho \frac{A_0}{M_0}$$

For R = 10 cm $\Rightarrow i = 9.3\times 10^{-24}\left(\frac{4}{3}\right)(.1)\,7.86\times 10^{3}\,\frac{\left(6.023\times 10^{26}\right)}{56}$

$= 1.05 \times 10^5$ Amperes

Thus, an ordinary piece of iron can have the same magnetic moment as a current loop of radius 10 cm of 10^5 Amperes current.

B. Magnetic Dipole Field

$$\bar{H} = \frac{\mu_0 m}{4\pi r^3 \mu_0}\left[2\cos\theta\, \bar{i}_r + \sin\theta\, \bar{i}_\theta\right]$$ (multiply top & bottom by μ_0)

Electric Dipole Field

$$\bar{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\, \bar{i}_r + \sin\theta\, \bar{i}_\theta\right]$$

Analogy

$$p \rightarrow \mu_0 m$$

$\bar{P} = N\bar{p} \Rightarrow \bar{M} = N\bar{m}$, N = # of magnetic dipoles / volume

Polarization Magnetization

XIV. Maxwell's EQS Equations with Magnetization

A. Analogy to Maxwell's EQS Equations with Polarization

EQS

$$\nabla \bullet \left(\varepsilon_0 \bar{E}\right) = \rho_u - \nabla \bullet \bar{P}$$

$\rho_p = -\nabla \bullet \bar{P}$ (Polarization or paired charge density)

$$\bar{n} \bullet \left[\varepsilon_0 \left(\bar{E}^a - \bar{E}^b\right)\right] = -\bar{n} \bullet \left[\bar{P}^a - \bar{P}^b\right] + \sigma_{su}$$

$$\sigma_{sp} = -\bar{n} \bullet \left[\bar{P}^a - \bar{P}^b\right]$$

MQS

$$\nabla \bullet \left(\mu_0 \bar{H}\right) = -\nabla \bullet \left(\mu_0 \bar{M}\right)$$

$\rho_m = -\nabla \bullet \left(\mu_0 \bar{M}\right)$ (magnetic charge density)

$$\bar{n} \bullet \left[\mu_0 \left(\bar{H}^a - \bar{H}^b\right)\right] = -\bar{n} \bullet \left[\mu_0 \left(\bar{M}^a - \bar{M}^b\right)\right]$$

$$\sigma_{sm} = -\bar{n} \bullet \left[\mu_0 \left(\bar{M}^a - \bar{M}^b\right)\right]$$

$$\nabla \times \bar{H} = \bar{J}$$

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t} \mu_0 \left(\bar{H} + \bar{M}\right)$$

B. <u>MQS Equations</u>

$\bar{B} = \mu_0 \left(\bar{H} + \bar{M}\right)$ Magnetic flux density $\bar{B}$ has units of Teslas (1 Tesla = 10,000 Gauss)

$$\nabla \bullet \bar{B} = 0$$

$$\bar{n} \bullet \left[\bar{B}^a - \bar{B}^a\right] = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}$$

$$v = \frac{d\lambda}{dt}, \quad \lambda = \int_S \bar{B} \bullet \overline{da} \text{ (total flux)}$$

XV. Magnetic Field Intensity along Axis of a Uniformly Magnetized Cylinder

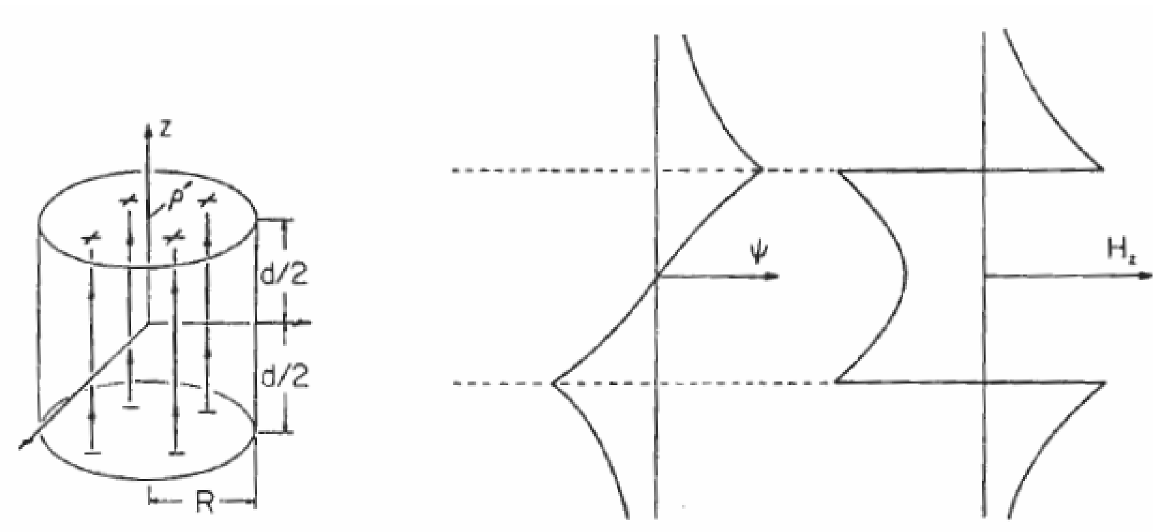


Figure 9.3.1 (a) Cylinder of circular cross-section uniformly magnetized in the direction of its axis. (b) Axial distribution of scalar magnetic potential and (c) axial magnetic field intensity. For these distributions, the cylinder length is assumed to be equal to its diameter.

From *Electromagnetic Fields and Energy* by Hermann A. Haus and James R. Melcher. Used with permission.

$$\sigma_{sm} = -n \bullet \mu_0 \left(\bar{M}^a - \bar{M}^b\right) \Rightarrow \sigma_{sm}\left(z = d/2\right) = \mu_0 M_0$$

$$\sigma_{sm}\left(z = -d/2\right) = -\mu_0 M_0$$

$$\nabla \times \bar{H} = \bar{J} = 0 \Rightarrow \bar{H} = -\nabla \Psi$$

$$\nabla \cdot \left(\mu_0 \overline{H}\right) = -\mu_0 \nabla^2 \Psi = \rho_m = -\nabla \cdot \left(\mu_0 \overline{M}\right)$$

$$\nabla^2 \Psi = {}^{-\rho_m}\!/_{\mu_0} \Rightarrow \Psi\left(\bar{r}\right) = \int_{V'} \frac{\rho_m\left(\bar{r}'\right) dV'}{4\pi\mu_0 \left|\bar{r} - \bar{r}'\right|}$$

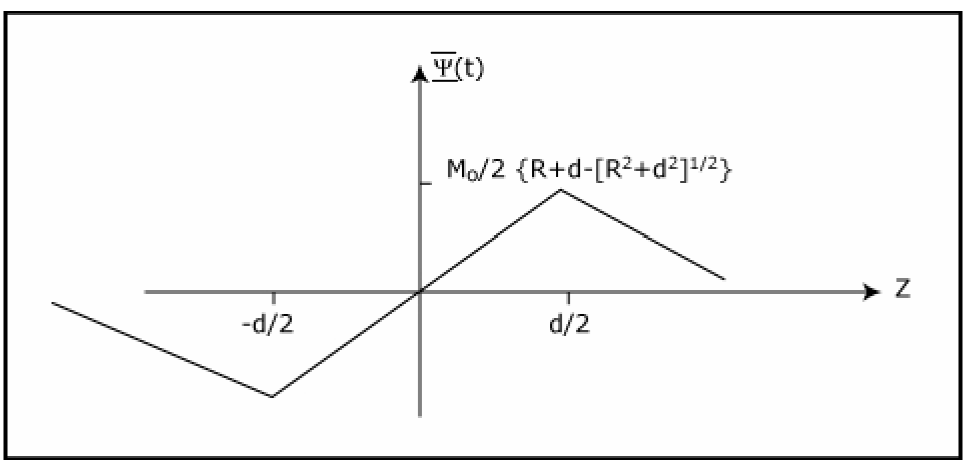
$$\Psi(z) = \int_{r'=0}^{R} \frac{\sigma_{sm}\left(z = {}^{d}\!/_{2}\right) 2\pi r' dr'}{4\pi\mu_0 \left|\bar{r} - \bar{r}'\right|} + \int_{r'=0}^{R} \frac{\sigma_{sm}\left(z = -{}^{d}\!/_{2}\right) 2\pi r' dr'}{4\pi\mu_0 \left|\bar{r} - \bar{r}'\right|}$$

$$= \int_{r'=0}^{R} \frac{\mu_0 M_o 2\pi r' dr'}{4\pi\mu_0 \left[r'^2 + \left(z - \frac{d}{2}\right)^2\right]^{1/2}} - \int_{r'=0}^{R} \frac{\mu_0 M_o 2\pi r' dr'}{4\pi\mu_0 \left[r'^2 + \left(z + \frac{d}{2}\right)^2\right]^{1/2}}$$

$$\int \frac{r' dr'}{\left[r'^2 + (z+a)^2\right]^{1/2}} = \left[r'^2 + (z+a)^2\right]^{1/2}$$

$$\Psi(z) = \frac{M_0}{2}\left\{\left[r'^2 + \left(z - \frac{d}{2}\right)^2\right]^{\frac{1}{2}}\Bigg|_{r'=0}^{R} - \left[r'^2 + \left(z + \frac{d}{2}\right)^2\right]^{\frac{1}{2}}\Bigg|_{r'=0}^{R}\right\}$$

$$= \frac{M_0}{2}\left\{\left[R^2 + \left(z - \frac{d}{2}\right)^2\right]^{1/2} - \left|z - \frac{d}{2}\right| - \left[R^2 + \left(z + \frac{d}{2}\right)^2\right]^{1/2} + \left|z + \frac{d}{2}\right|\right\}$$



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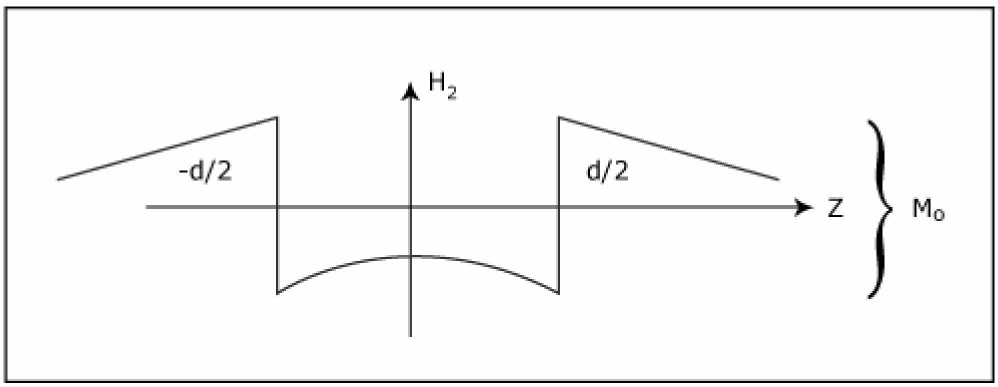
$$
H_z = \frac{-\partial \Psi}{\partial z} = \begin{cases} \dfrac{-M_0}{2}\left\{ \dfrac{z-\frac{d}{2}}{\left[R^2+\left(z-\frac{d}{2}\right)^2\right]^{1/2}} - \dfrac{\left(z+\frac{d}{2}\right)}{\left[R^2+\left(z+\frac{d}{2}\right)^2\right]^{1/2}} \right\} & |z| > \dfrac{d}{2} \\ \\ \dfrac{-M_0}{2}\left\{ \dfrac{z-\frac{d}{2}}{\left[R^2+\left(z-\frac{d}{2}\right)^2\right]^{1/2}} - \dfrac{\left(z+\frac{d}{2}\right)}{\left[R^2+\left(z+\frac{d}{2}\right)^2\right]^{1/2}} + 2 \right\} & -\dfrac{d}{2} < z < \dfrac{d}{2} \end{cases}
$$



XVI. Toroidal Coil

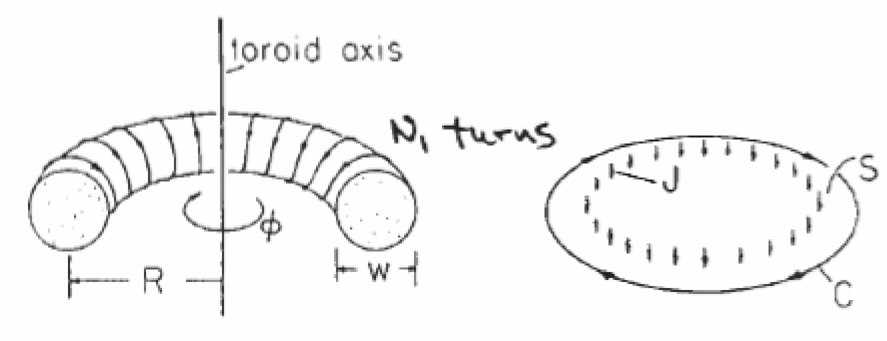


Figure 9.4.1 Toroidal coil with donut-shaped magnetizable core.

Figure 9.4.2 Surface S enclosed by contour C used with Ampère's integral law to determine **H** in the coil shown in Figure 9.4.1.

$$\oint_C \overline{H} \cdot \overline{dl} = H_\phi \, 2\pi r = N_1 i \Rightarrow H_\phi = \frac{N_1 i}{2\pi r} \approx \frac{N_1 i}{2\pi R}$$

$$\Phi \approx B \; \frac{\pi w^2}{4}$$

$$\lambda = N_2 \, \Phi = N_2 \, B \frac{\pi w^2}{4}$$

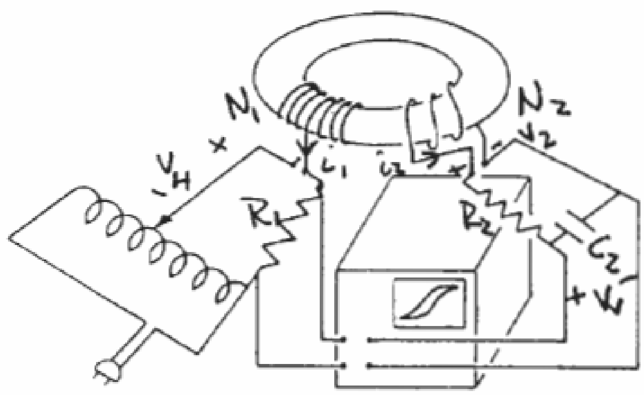


Figure 9.4.3 Demonstration in which the $B - H$ curve is traced out in the sinusoidal steady state.

$$V_H = i_1 R_1 = R_1 \frac{H_\phi \, 2\pi R}{N_1} \qquad (V_H = \text{Horizontal voltage to oscilloscope})$$

$$v_2 = \frac{d\lambda_2}{dt} = i_2 R_2 + V_v = V_v + R_2 C_2 \frac{dV_v}{dt}$$

$$\text{If } R_2 >> \frac{1}{C_2 \omega} \Rightarrow \frac{d\lambda_2}{dt} \approx R_2 C_2 \frac{dV_v}{dt} \Rightarrow \lambda_2 = R_2 C_2 V_v \qquad (V_v = \text{Vertical voltage to oscilloscope})$$

$$= \frac{\pi w^2}{4} N_2 B$$

$$V_v = \frac{1}{R_2 C_2} \frac{\pi w^2}{4} N_2 B$$

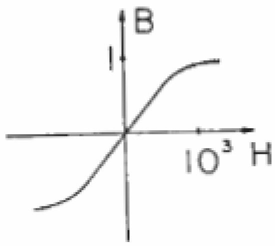


Figure 9.4.4 Typical magnetization curve without hysteresis. For typical ferromagnetic solids, the saturation flux density is in the range of 1–2 Tesla. For ferromagnetic domains suspended in a liquid, it is .02–.04 Tesla.

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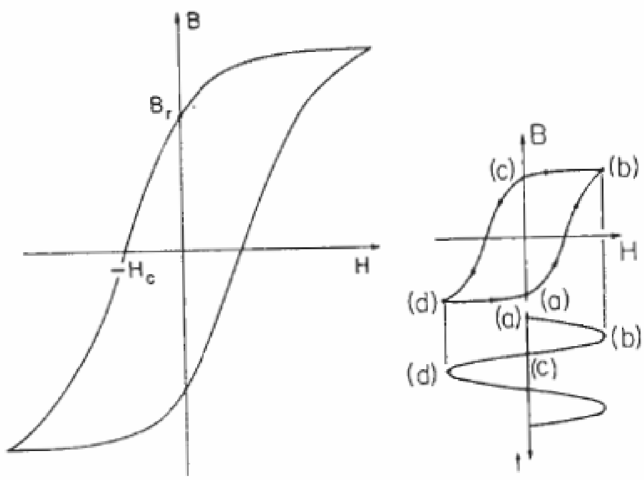

Figure 9.4.6 Magnetization characteristic for material showing hysteresis with typical values of B_r and H_c given in Table 9.4.2. The curve is obtained after many cycles of sinusoidal excitation in apparatus such as that of Figure 9.4.3. The trajectory is traced out in response to a sinusoidal current, as shown by the inset.

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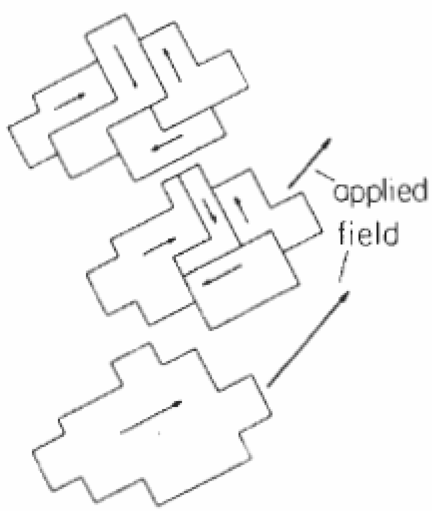

Figure 9.4.5 Polycrystalline ferromagnetic material viewed at the domain level. In the absence of an applied magnetic field, the domain moments tend to cancel. (This presumes that the material has not been left in a magnetized state by a previously applied field.) As a field is applied, the domain walls shift, giving rise to a net magnetization. In ideal materials, saturation results as all of the domains combine into one. In materials used for bulk fabrication of transformers, imperfections prevent the realization of this state.

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XVII. Magnetic Circuits

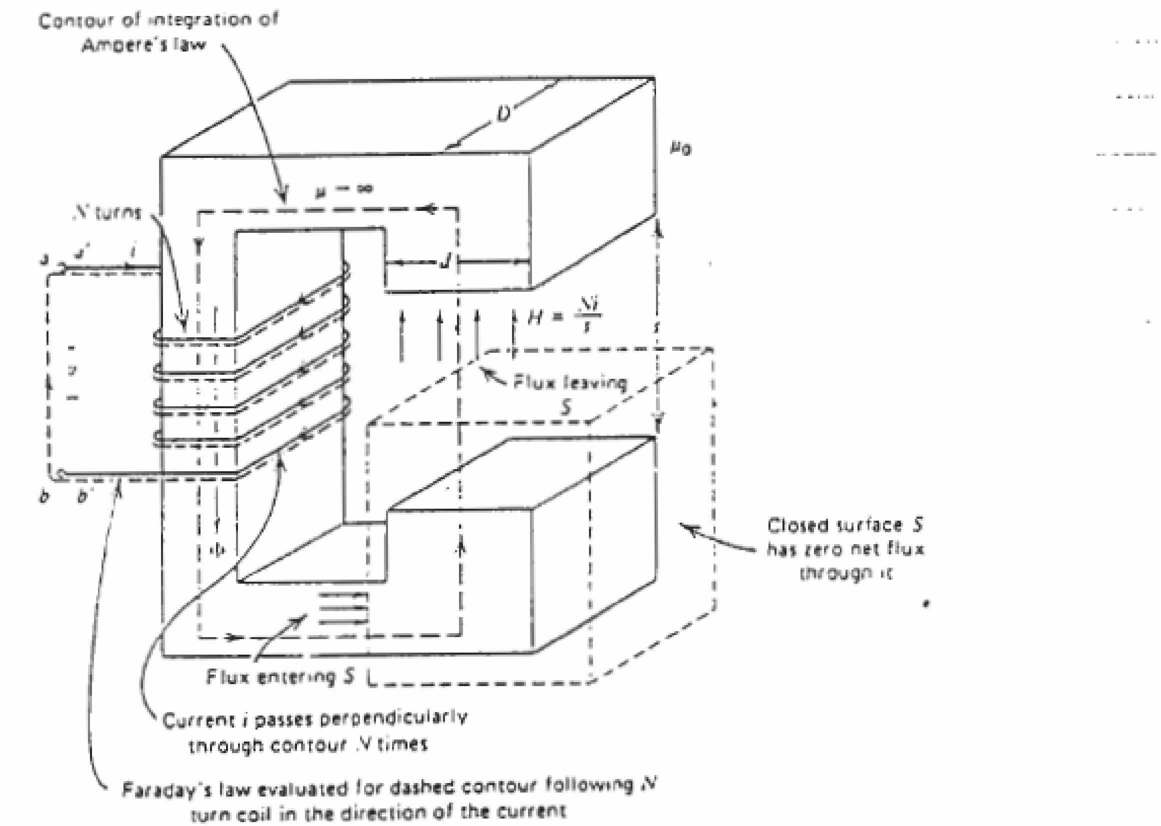

Figure 6-8 The magnetic field is zero within an infinitely permeable magnetic core and is constant in the air gap if we neglect fringing. The flux through the air gap is constant at every cross section of the magnetic circuit and links the $\mathcal{N}$ turn coil $\mathcal{N}$ times.

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In iron core:

$$\lim_{\mu \to \infty} \bar{B} = \mu \bar{H} \Rightarrow \begin{cases} \bar{H} = 0 \\ \bar{B} \text{ finite} \end{cases}$$

$$\oint \bar{H} \cdot \overline{dl} = Hs = Ni \Rightarrow H = \frac{Ni}{s}$$

$$\Phi = \mu_0 \; H \; Dd = \frac{\mu_0 \; Dd \; N}{s} i$$

$$\oint_S \bar{B} \cdot \overline{da} = 0$$

$$\lambda = N \Phi = \frac{\mu_0 \; Dd}{s} N^2 i \Rightarrow L = \frac{\lambda}{i} = \frac{\mu_0 \; Dd}{s} N^2$$

XVIII. Reluctance

$$\mathcal{R} = \frac{Ni}{\Phi} = \frac{s}{\mu_0 \ Dd} = \frac{(\text{length})}{(\text{permeability})(\text{cross}-\text{sectional area})}$$

[Reluctance, analogous to resistance]

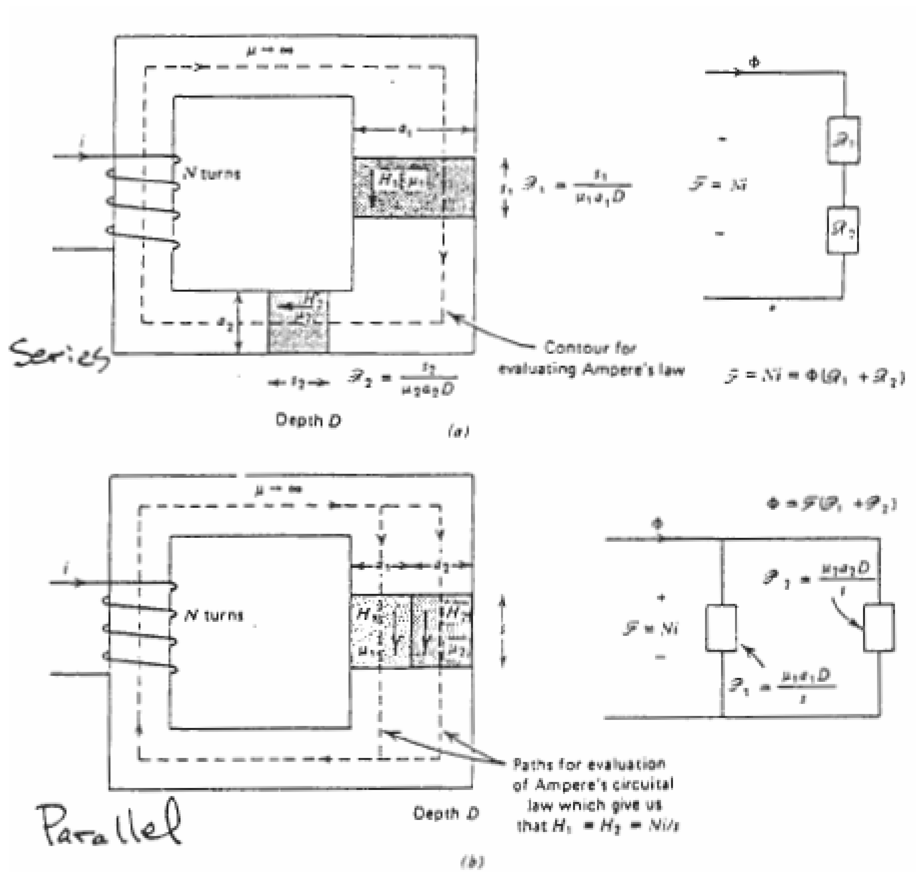

Figure 6-11 Magnetic circuits are most easily analyzed from a circuit approach where (a) reluctances in series add and (b) permeances in parallel add.

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A. Reluctances In Series

$$\mathcal{R}_1 = \frac{s_1}{\mu_1 \, a_1 \, D}\ , \quad \mathcal{R}_2 = \frac{s_2}{\mu_2 \, a_2 \, D}$$

$$\Phi = \frac{Ni}{\mathcal{R}_1 + \mathcal{R}_2}$$

$$\oint_C \overline{H} \bullet \overline{dl} = H_1 \, s_1 + H_2 \, s_2 = Ni$$

$$\Phi = \mu_1 H_1 a_1 D = \mu_2 H_2 a_2 D$$

$$H_1 = \frac{\mu_2 a_2 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1} \; ; \quad H_2 = \frac{\mu_1 a_1 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1}$$

B. Reluctances In Parallel

$$\oint_C \overline{H} \cdot \overline{dl} = H_1 s = H_2 s = Ni \Rightarrow H_1 = H_2 = \frac{Ni}{s}$$

$$\Phi = \left(\mu_1 H_1 a_1 + \mu_2 H_2 a_2\right) D = \frac{Ni\left(\mathcal{R}_1 + \mathcal{R}_2\right)}{\mathcal{R}_1 \mathcal{R}_2} = Ni\left(\mathcal{P}_1 + \mathcal{P}_2\right)$$

$$\mathcal{P}_1 = \frac{1}{\mathcal{R}_1} \; ; \quad \mathcal{P}_2 = \frac{1}{\mathcal{R}_2}$$

$\mathcal{P} = \frac{1}{\mathcal{R}}$ [Permeances, analogous to Conductance]

XIX. Transformers
(Ideal)

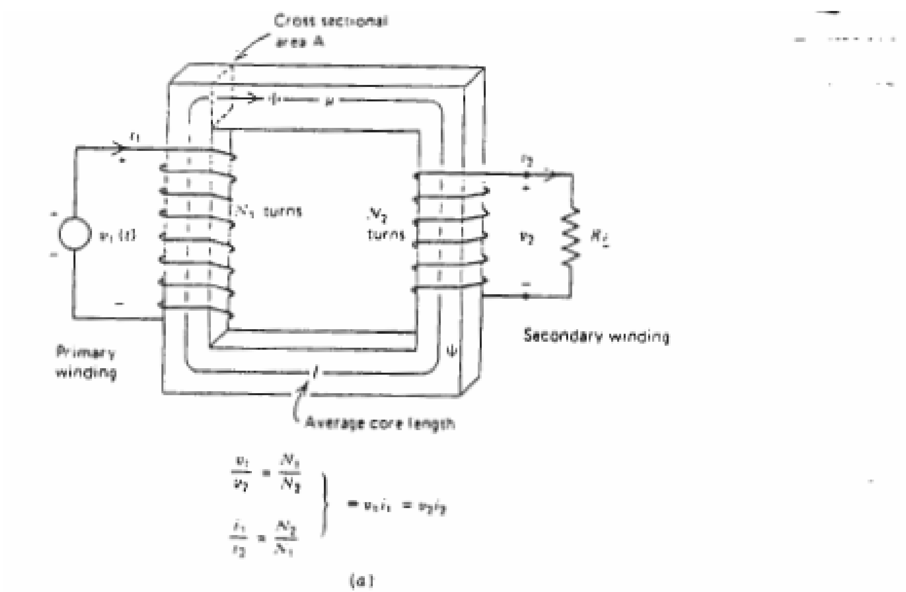

Figure 6-13 (a) An ideal transformer relates primary and secondary voltages by the ratio of turns while the currents are in the inverse ratio so that the input power equals the output power. The H field is zero within the infinitely permeable core. (b) In a real transformer the nonlinear B-H hysteresis loop causes a nonlinear primary current i_1 with an open circuited secondary ($i_2 = 0$) even though the imposed sinusoidal voltage $v_1 = V_0 \cos \omega t$ fixes the flux to be sinusoidal. (c) A more complete transformer equivalent circuit.

A. Voltage/Current Relationships

$$\Phi = \frac{N_1 i_1 - N_2 i_2}{\mathcal{R}}\ ; \qquad \mathcal{R} = \frac{l}{\mu A}$$

Another way: $\oint_C H \bullet dl = Hl = N_1 i_1 - N_2 i_2$

$$H = \frac{N_1 i_1 - N_2 i_2}{l}$$

$$\Phi = \mu H A = \frac{\mu A}{l}\left(N_1 i_1 - N_2 i_2\right) = \frac{N_1 i_1 - N_2 i_2}{R}$$

$$\lambda_1 = N_1 \Phi = \frac{\mu A}{l}\left(N_1^2 i_1 - N_1 N_2 i_2\right) = L_1 i_1 - M i_2$$

$$\lambda_2 = N_2 \Phi = \frac{\mu A}{l}\left(N_1 N_2 i_1 - N_2^2 i_2\right) = -M i_1 + L_2 i_2$$

$$L_1 = N_1^2 L_0\,, \quad L_2 = N_2^2 L_0\,, \quad M = N_1 N_2 L_0\,, \quad L_0 = \frac{\mu A}{l} = 1/\mathcal{R}$$

$$M = \left[L_1 L_2\right]^{1/2}$$

$$v_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} - M\frac{di_2}{dt} = N_1 L_0\left[N_1 \frac{di_1}{dt} - N_2 \frac{di_2}{dt}\right]$$

$$v_2 = \frac{d\lambda_2}{dt} = +M\frac{di_1}{dt} - L_2 \frac{di_2}{dt} = N_2 L_0\left[+N_1 \frac{di_1}{dt} - N_2 \frac{di_2}{dt}\right]$$

$$\frac{v_1}{v_2} = \frac{N_1}{N_2}$$

$$\lim_{\mu\to\infty} H \Rightarrow 0 \Rightarrow N_1 i_1 = N_2 i_2 \Rightarrow \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

$$\frac{v_1 i_1}{v_2 i_2} = 1$$

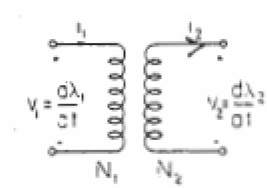

Figure 9.7.6 Circuit representation of a transformer as defined by the terminal relations of (12) or of an ideal transformer as defined by (13).