6.013 - Electromagnetics and Applications Fall 2005

Lecture 10 - Transmission Lines

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I. Transmission Line Equations

A. Parallel Plate Transmission Line

$\bar{E}$ must be perpendicular to the electrodes and $\bar{H}$ must be tangential, so

$$\bar{E} = E_x(z,t)\bar{i}_x$$
$$\bar{H} = H_y(z,t)\bar{i}_y$$

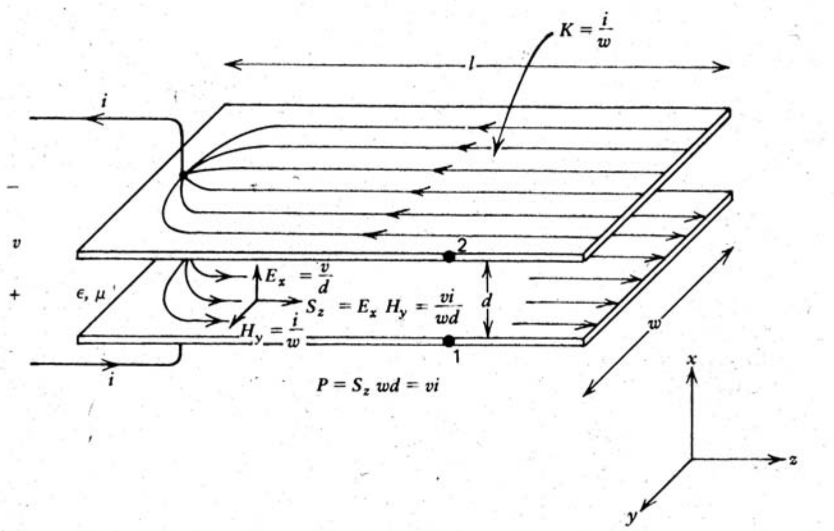

Figure 8-1 The simplest transmission line consists of two parallel perfectly conducting plates a small distance d apart.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \Rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$
$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \frac{\partial H_y}{\partial z} = -\epsilon \frac{\partial E_x}{\partial t}$$

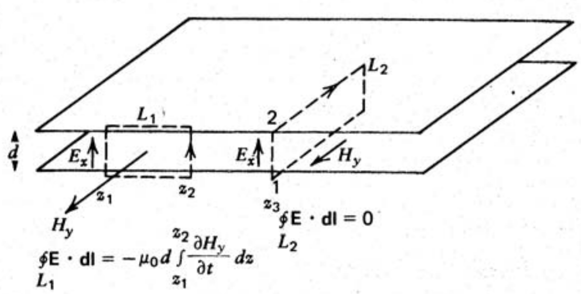

Figure 8-2 The potential difference measured between any two arbitrary points at different positions z_1 and z_2 on the transmission line is not unique—the line integral L_1 of the electric field is nonzero since the contour has magnetic flux passing through it. If the contour L_2 lies within a plane of constant z such as at z_3, no magnetic flux passes through it so that the voltage difference between the two electrodes at the same value of z is unique.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$v(z,t) = \int_{\substack{1 \\ z = \text{constant}}}^{2} \bar{E} \cdot d\bar{l} = E_x(z,t) d$$

$$i(z,t) = K_z(z,t) w = H_y(z,t) w$$

$$\frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t} \quad L = \frac{\mu d}{w} \text{ henries / meter} \quad \text{Inductance per unit length}$$

$$\frac{\partial i}{\partial z} = -C \frac{\partial v}{\partial t} \quad C = \frac{\epsilon w}{d} \text{ farads / meter} \quad \text{Capacitance per unit length}$$

$$LC = \frac{\mu \cancel{d}}{\cancel{w}} \frac{\epsilon \cancel{w}}{\cancel{d}} = \mu\epsilon = \frac{1}{c^2}$$

B. Transmission Line Structures

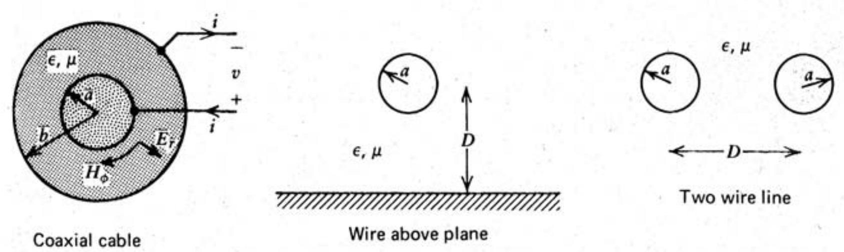

Figure 8-3 Various types of simple transmission lines.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

EXAMPLE 8-1 THE COAXIAL TRANSMISSION LINE

Consider the coaxial transmission line shown in Figure 8-3 composed of two perfectly conducting concentric cylinders of radii a and b enclosing a linear medium with permittivity ε and permeability μ. We solve for the transverse dependence of the fields as if the problem were static, independent of time. If the voltage difference between cylinders is v with the inner cylinder carrying a total current i the static fields are

$$E_r = \frac{v}{r \ln (b/a)}, \qquad H_\phi = \frac{i}{2\pi r}$$

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The surface charge per unit length q and magnetic flux per unit length λ are

$$q = \varepsilon E_r(r = a) 2\pi a = \frac{2\pi\varepsilon v}{\ln(b/a)}$$

$$\lambda = \int_a^b \mu H_\phi \, dr = \frac{\mu i}{2\pi} \ln \frac{b}{a}$$

so that the capacitance and inductance per unit length of this structure are

$$C = \frac{q}{v} = \frac{2\pi\varepsilon}{\ln(b/a)}, \qquad L = \frac{\lambda}{i} = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

where we note that as required

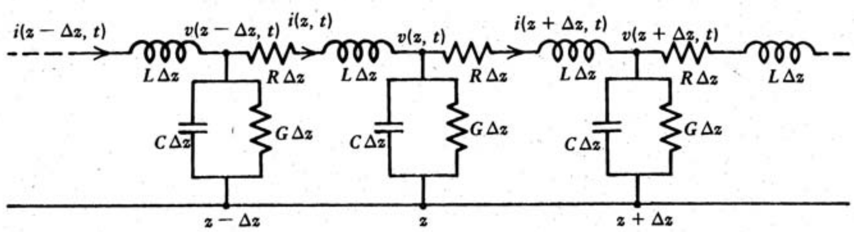
$$LC = \varepsilon\mu$$

Substituting E_r and H_ϕ into (12) yields the following transmission line equations:

$$\frac{\partial E_r}{\partial z} = -\mu \frac{\partial H_\phi}{\partial t} \Rightarrow \frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t}$$

$$\frac{\partial H_\phi}{\partial z} = -\varepsilon \frac{\partial E_r}{\partial t} \Rightarrow \frac{\partial i}{\partial z} = -C \frac{\partial v}{\partial t}$$

C. Distributed Circuit Representation with Losses



$$v(z,t) - v(z+\Delta z, t) = L\,\Delta z \frac{\partial}{\partial t} i(z+\Delta z, t) + i(z+\Delta z, t) R\,\Delta z$$

$$i(z,t) - i(z+\Delta z, t) = C\,\Delta z \frac{\partial}{\partial t} v(z,t) + G\,\Delta z\, v(z,t)$$

Figure 8-5 Distributed circuit model of a transmission line including small series and shunt resistive losses.

$$i(z,t) - i(z+\Delta z, t) = C\Delta z \frac{\partial v(z,t)}{\partial t} + G\Delta z\, v(z,t)$$

$$v(z,t) - v(z+\Delta z, t) = L\Delta z \frac{\partial i(z+\Delta z, t)}{\partial t} + i(z+\Delta z, t) R\Delta z$$

$$\lim_{\Delta z \to 0} \frac{i(z+\Delta z, t) - i(z,t)}{\Delta z} = \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} - Gv$$

$$\lim_{\Delta z \to 0} \frac{v(z+\Delta z, t) - v(z,t)}{\Delta z} = \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} - iR$$

R is the series resistance per unit length, measured in ohms/meter, and G is the shunt conductance per unit length, measured in siemens/meter.

If the line is lossless ($R = G = 0$), we have the Telegrapher's equations:

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}$$
$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

Including loss, Poynting's theorem for the circuit equivalent form is:

$$v \cdot \left| \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} - Gv \right.$$
$$i \cdot \left| \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} - iR \right.$$
$$\text{Add: } v\frac{\partial i}{\partial z} + i\frac{\partial v}{\partial z} = \frac{\partial (vi)}{\partial z} = -\frac{\partial}{\partial t}\left[\frac{1}{2}Cv^2 + \frac{1}{2}Li^2\right] - Gv^2 - i^2R$$

D. Wave Equation (Lossless, $R = 0, G = 0$)

$$\frac{\partial}{\partial t}\left| \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} \right. \quad \Rightarrow \quad \frac{\partial^2 i}{\partial z \partial t} = -C\frac{\partial^2 v}{\partial t^2}$$
$$\frac{\partial}{\partial z}\left| \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} \right. \quad \quad \frac{\partial^2 i}{\partial z \partial t} = -\frac{1}{L}\frac{\partial^2 v}{\partial z^2}$$
$$-\frac{1}{L}\frac{\partial^2 v}{\partial z^2} = -C\frac{\partial^2 v}{\partial t^2} \quad \Rightarrow \quad \underbrace{\frac{\partial^2 v}{\partial z^2} = LC\frac{\partial^2 v}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}}_{\text{Wave equation}}$$

II. Sinusoidal Steady State

A. Complex Amplitude Notation

$$v(z,t) = \text{Re}\left[\hat{v}(z)e^{j\omega t}\right]$$
$$i(z,t) = \text{Re}\left[\hat{i}(z)e^{j\omega t}\right]$$

Substitute into the wave equation:

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \Rightarrow \frac{d^2\hat{v}}{dz^2} = -\frac{\omega^2}{c^2}\hat{v}(z), \text{ let } k = \frac{\omega}{c}$$
$$\frac{d^2\hat{v}}{dz^2} + k^2\hat{v} = 0 \Rightarrow \hat{v}(z) = \hat{V}_+e^{-jkz} + \hat{V}_-e^{+jkz}$$
$$\frac{d\hat{v}}{dz} = -Lj\omega\hat{i} \Rightarrow \hat{i}(z) = -\frac{1}{L\cancel{j}\omega}\left(-\cancel{j}k\hat{V}_+e^{-jkz} + \cancel{j}k\hat{V}_-e^{+jkz}\right)$$
$$\frac{k}{\omega} = \frac{\cancel{\omega}}{c\cancel{\omega}} = \sqrt{LC} \Rightarrow \frac{k}{\omega L} = \frac{\sqrt{LC}}{L} = \sqrt{\frac{C}{L}} = Y_0 \text{ is the Line Admittance}$$
$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L}{C}} \text{ is the Line Impedance}$$
$$\hat{i}(z) = Y_0\left(\hat{V}_+e^{-jkz} - \hat{V}_-e^{+jkz}\right)$$

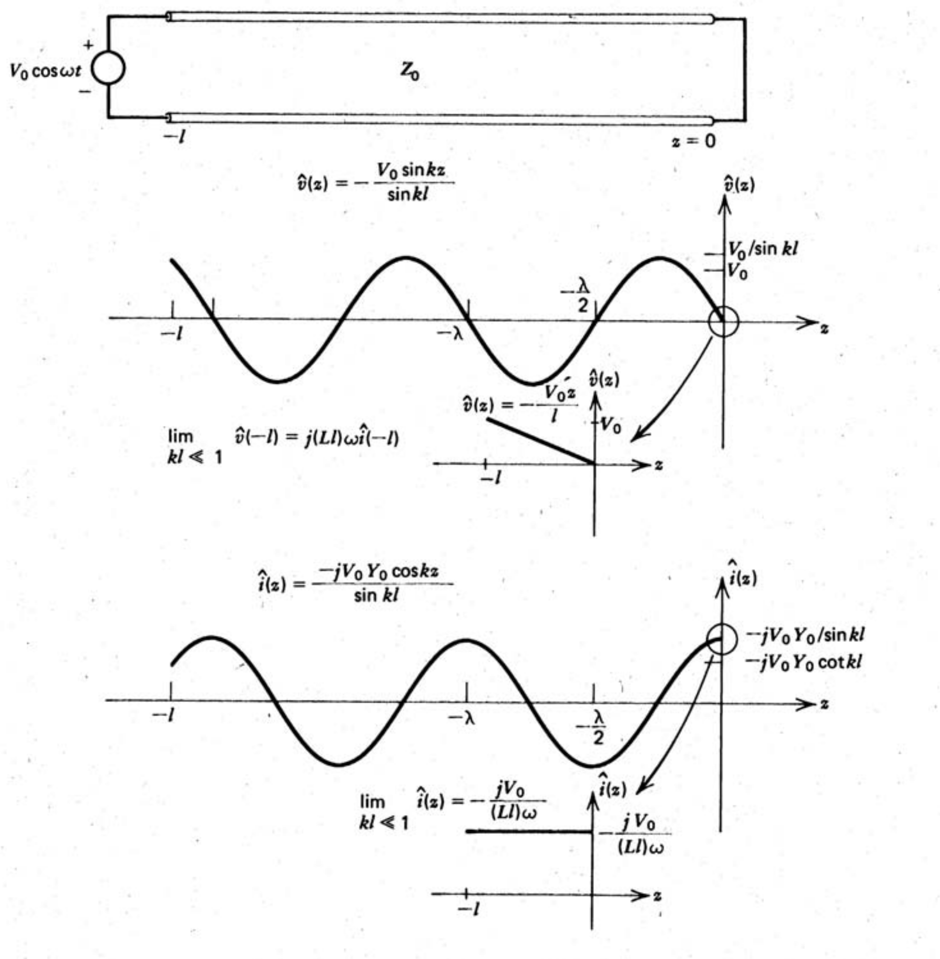
$$\hat{v}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz}$$

$$v(z,t) = \mathrm{Re}\left[\hat{V}_+ e^{j(\omega t - kz)} + \hat{V}_- e^{j(\omega t + kz)}\right]$$

$$i(z,t) = \mathrm{Re}\, Y_0 \left[\hat{V}_+ e^{j(\omega t - kz)} - \hat{V}_- e^{j(\omega t + kz)}\right]$$

$$k = \frac{\omega}{c} = \omega\sqrt{LC} = \omega\sqrt{\epsilon\mu}$$

B. Short Circuited Line ($v(z=0,t) = 0, v(z=-l,t) = V_0\cos(\omega t)$)



(a)

Figure 8-15 The voltage and current distributions on a *(a)* short circuited and *(b)* open circuited transmission line excited by sinusoidal voltage sources at $z = -l$. If the lines are much shorter than a wavelength, they act like reactive circuit elements. *(c)* As the frequency is raised, the impedance reflected back as a function of z can look capacitive or inductive making the transition through open or short circuits.

$$\hat{v}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz} \Rightarrow \hat{v}(z=0) = 0 = \hat{V}_+ + \hat{V}_- \Rightarrow \hat{V}_+ = -\hat{V}_-$$

$$\hat{v}(z=-l) = V_0 = \hat{V}_+ e^{+jkl} + \hat{V}_- e^{-jkl} = \hat{V}_+\left(e^{jkl} - e^{-jkl}\right)$$

$$= 2j\sin(kl)\hat{V}_+$$

$$\hat{V}_+ = -\hat{V}_- = \frac{V_0}{2j\sin(kl)}$$

$$\hat{v}(z) = \frac{V_0}{2j\sin(kl)}\left(e^{-jkz} - e^{+jkz}\right) = \frac{V_0(-2j)\sin(kz)}{2j\sin(kl)}$$

$$= -\frac{V_0\sin(kz)}{\sin(kl)}$$

$$\hat{i}(z) = Y_0\left(\hat{V}_+e^{-jkz} - \hat{V}_-e^{jkz}\right) = \frac{Y_0V_0}{2j\sin(kl)}\left(e^{-jkz} + e^{+jkz}\right)$$

$$= \frac{\not{2}Y_0V_0\cos(kz)}{\not{2}j\sin(kl)}$$

$$= -\frac{jY_0V_0\cos(kz)}{\sin(kl)}$$

$$v(z,t) = \text{Re}\left[\hat{v}(z)e^{j\omega t}\right] = \text{Re}\left[-\frac{V_0\sin(kz)}{\sin(kl)}e^{j\omega t}\right] = -\frac{V_0\sin(kz)\cos(\omega t)}{\sin(kl)}$$

$$i(z,t) = \text{Re}\left[\hat{i}(z)e^{j\omega t}\right] = \text{Re}\left[-\frac{jY_0V_0\cos(kz)}{\sin(kl)}e^{j\omega t}\right] = \frac{Y_0V_0\cos(kz)\sin(\omega t)}{\sin(kl)}$$

We have resonance when $\sin(kl) = 0 \Rightarrow kl = n\pi = \frac{\omega l}{c} \Rightarrow \omega = \omega_n \equiv \frac{n\pi c}{l}, n = 1, 2, 3, \ldots$

Complex impedance: $Z(z) = \frac{\hat{v}(z)}{\hat{i}(z)} = -jZ_0\tan(kz)$

$$Z(z=-l) = +jZ_0\tan(kl)$$

In the following, take $n = 1, 2, 3, \ldots$:

$kl = n\pi$	$Z(z=-l) = 0$	short circuit
$kl = (2n-1)\frac{\pi}{2}$	$Z(z=-l) = \infty$	open circuit
$(n-1)\pi < kl < (2n-1)\frac{\pi}{2}$	$Z(z=-l) = +jX, X > 0$	(positive reactance, inductive)
$(n-\frac{1}{2})\pi < kl < n\pi$	$Z(z=-l) = -jX, X > 0$	(negative reactance, capacitive)
$kl \ll 1 \Rightarrow$	$Z(z) = -jZ_0k$	
	$= -j\sqrt{\frac{L}{\not{C}}}\,\omega\sqrt{L\not{C}}$	
	$= -jLZ$	
	$Z(z=-l) = j(Ll)$ inductive	
$\lvert kz\rvert \ll 1$	$v(z,t) = -\frac{V_0 z}{l}\cos(\omega t) \Rightarrow$	$v(z=-l,t) = V_0\cos(\omega t)$
		$= (Ll)\frac{di}{dt}(z=-l,t)$
	$i(z,t) = \frac{V_0Y_0}{kl}\sin(\omega t)$	$i(z=-l,t) = \frac{V_0\sin(\omega t)}{(Ll)\omega}$

C. Open Circuited Line ($i(z=0,t)=0$)

$$v(z=-l,t) = V_0 \sin(\omega t)$$

$$\hat{i}(z) = Y_0\left[\hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz}\right] \Rightarrow \hat{i}(z=0) = 0 = Y_0\left[\hat{V}_+ - \hat{V}_-\right] \Rightarrow \hat{V}_+ = \hat{V}_-$$

$$\hat{v}(z=-l) = -jV_0 = \hat{V}_+ e^{+jkl} + \hat{V}_- e^{-jkl} = \hat{V}_+\left(e^{jkl} + e^{-jkl}\right) = 2\hat{V}_+\cos(kl)$$

$$\hat{V}_+ = \hat{V}_- = -\frac{jV_0}{2\cos(kl)}$$

$$\begin{aligned}
\hat{v}(z) &= -\frac{jV_0}{2\cos(kl)}\left(e^{-jkz} + e^{+jkz}\right) \\
&= -\frac{jV_0 \cdot \cancel{2}\cos(kz)}{\cancel{2}\cos(kl)} \\
&= -\frac{jV_0\cos(kz)}{\cos(kl)} \\
\hat{i}(z) &= -\frac{jY_0V_0}{2\cos(kl)}\left(e^{-jkz} - e^{+jkz}\right) \\
&= \frac{(-jY_0V_0)(-2j)\sin(kz)}{2\cos(kl)} \\
&= -\frac{Y_0V_0\sin(kz)}{\cos(kl)}
\end{aligned}$$

$$v(z,t) = \mathrm{Re}\left[\hat{v}(z)e^{j\omega t}\right] = \frac{V_0\cos(kz)}{\cos(kl)}\sin(\omega t)$$

$$i(z,t) = \mathrm{Re}\left[\hat{i}(z)e^{j\omega t}\right] = \frac{-V_0Y_0}{\cos(kl)}\sin(kz)\cos(\omega t)$$

$$\text{Resonance: } \cos(kl) = 0 \Rightarrow (kl) = (2n-1)\frac{\pi}{2}, n = 1,2,3,\ldots$$

$$\omega_n = \frac{(2n-1)\frac{\pi}{2}}{2l}$$

Complex Impedance

$$Z(z) = \frac{\hat{v}(z)}{\hat{i}(z)} = Z_0 j\cot(kz)$$

$$Z(z=-l) = -jZ_0\cot(kl)$$

$$kl \ll 1 \Rightarrow v(z,t) = V_0\sin(\omega t)$$

$$i(z,t) = -V_0Y_0kz\cos(\omega t)$$

$$i(z=-l,t) = (Cl)\omega V_0\cos(\omega t) = (Cl)\frac{dv}{dt}(z=-l,t)$$

Open circuited line

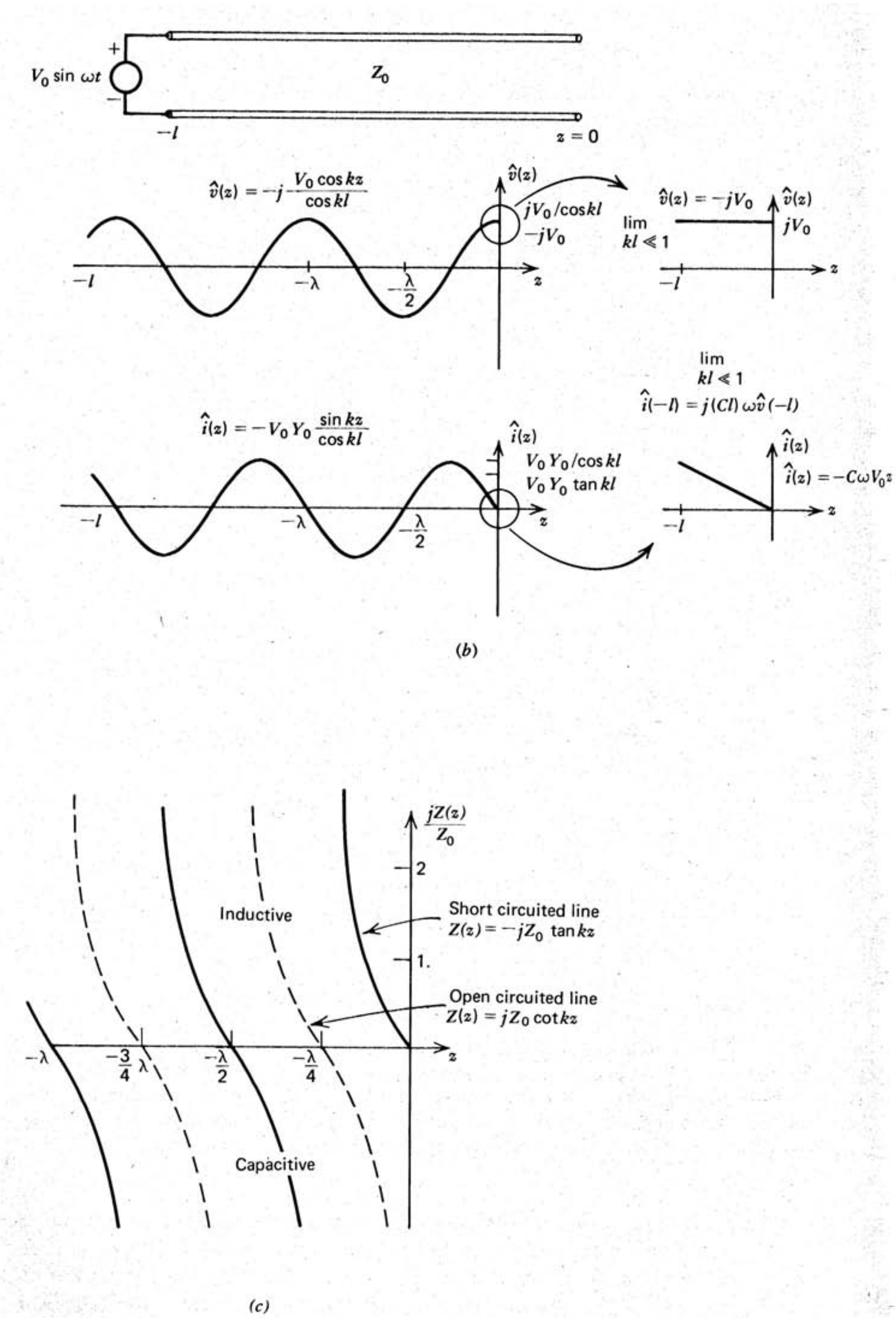


Figure 8-15

Impedance for short and open circuited wires