Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.013 Electromagnetics and Applications

Problem Set #6 | Issued: 10/12/05
Fall Term 2005 | Due: 10/26/05

Reading Assignment:

Quiz 1: Thursday, Oct. 20, 2005 in lecture, 10-11AM. Covers material up to and including P.S. #5.

Problem 6.1

An electric field is present within a plasma of dielectric permittivity ε with conduction constituent relation

$$\frac{\partial \mathbf{J}_f}{\partial t} = \omega_p^2 \varepsilon \mathbf{E} \quad , \quad \text{where} \quad \omega_p^2 = \frac{q^2 n}{m\varepsilon}$$

with q, n and m being the charge, number density (number per unit volume) and mass of each charge carrier.

(a) Poynting's theorem is

$$\nabla \bullet \mathbf{S} + \frac{\partial w_{\text{EM}}}{\partial t} = -\mathbf{E} \bullet \mathbf{J_f}$$

For the plasma medium, $\mathbf{E} \bullet \mathbf{J_f}$, can be written as

$$\mathbf{E} \bullet \mathbf{J_f} = \frac{\partial \mathrm{w}_k}{\partial \mathrm{t}} \quad .$$

What is w_k?

(b) What is the velocity $\boldsymbol{v}$ of the charge carriers in terms of the current density $\mathbf{J}_f$ and parameters q, n and m defined above?

(c) Write w_k of part (a) in terms of $\boldsymbol{v}$, q, n, and m. What kind of energy density is w_k?

(d) Assuming that all fields vary sinusoidally with time as:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\left[\hat{\boldsymbol{E}}(\boldsymbol{r}) e^{j\omega t}\right]$$

write Maxwell's equations in complex amplitude form with the plasma constitutive law.

(e) Reduce the complex Poynting theorem from the usual form

$$\nabla \bullet \left[\frac{1}{2}\hat{\mathbf{E}}(r) \times \hat{\mathbf{H}}^*(r)\right] + 2j\omega < w_{EM} > = -\frac{1}{2}\hat{\mathbf{E}} \bullet \hat{\mathbf{J}}_\mathbf{f}^*$$

to

$$\nabla \bullet \left[\frac{1}{2} \hat{\mathbf{E}}(r) \times \hat{\mathbf{H}}^*(r) \right] + 2j\omega(< w_{EM} > + < w_k >) = 0$$

What are $< w_{EM} > and < w_k >$?

(f) Show that

$$< w_{EM} > + < w_k > = \frac{1}{4}\mu |\mathbf{H}|^2 - \frac{1}{4}\varepsilon(\omega)|\mathbf{E}|^2$$

What is $\varepsilon(\omega)$ and compare to the results from Problem 5.3b?

Problem 6.2

A TEM wave (E_x, H_y) propagates in a medium whose dielectric permittivity and magnetic permeability are functions of $z, \varepsilon(z)$ and $\mu(z)$.

(a) Write down Maxwell's equations and obtain a single partial differential equation in H_y.

(b) Consider the idealized case where $\varepsilon(z) = \varepsilon_a e^{+\alpha z}$ and $\mu(z) = \mu_a e^{-\alpha z}$. Show that the equation of (a) for H_y reduces to a linear partial differential equation with constant coefficients of the form

$$\frac{\partial^2 H_y}{\partial z^2} - \beta \frac{\partial H_y}{\partial z} - \gamma \frac{\partial^2 H_y}{\partial t^2} = 0$$

What are β and γ?

(c) Infinite magnetic permeability regions with zero magnetic field extend for z<0 and z>d. A current sheet $\text{Re}\left[\overline{i}_x K_0 e^{j\omega t} \right]$ is placed at $z = 0$. Take the magnetic field of the form

$$\boldsymbol{H} = \text{Re}[\overline{i}_y \hat{\text{H}}_y e^{(j\omega t - \kappa z)}]$$

and find values of κ that satisfy the governing equation in (b) for 0<z<d.

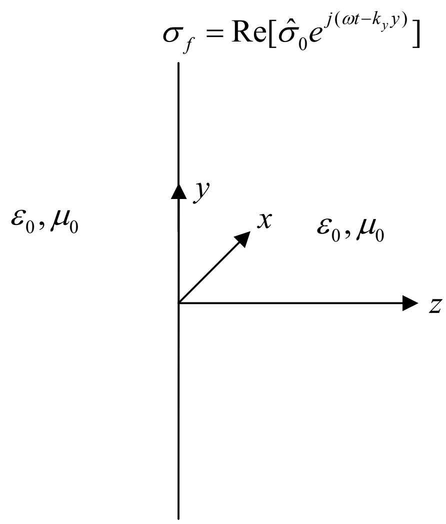
(d) What are the boundary conditions on $\boldsymbol{H}$?

(e) Superpose the solutions found in (c) and find $\boldsymbol{H}$ that satisfies the boundary conditions of (d).

(f) What is the electric field for 0<z<d?

Problem 6.3

A sheet of surface charge with charge density $\sigma_f = \mathrm{Re}[\hat{\sigma}_0 e^{j(\omega t - k_y y)}]$ is placed in free space (ε_0, μ_0) at $z = 0$.



The complex magnetic field in each region is of the form

$$\hat{\bar{H}} = \begin{cases} \hat{H}_1 e^{-j(k_y y + k_z z)} \bar{i}_x & z > 0 \\ \hat{H}_2 e^{-j(k_y y - k_z z)} \bar{i}_x & z < 0 \end{cases}$$

(a) What is k_z?

(b) What is the complex electric field for $z < 0$ and $z > 0$ in terms of $\hat{H}_1, \hat{H}_2, k_y, k_z$ and ω?

(c) Using the boundary conditions at $z = 0$, what are $\hat{H}_1$ and $\hat{H}_2$?

(d) For what range of the frequency will the waves for $z < 0$ and $z > 0$ be evanescent?

(e) What surface current flows on the charge sheet at $z = 0$?

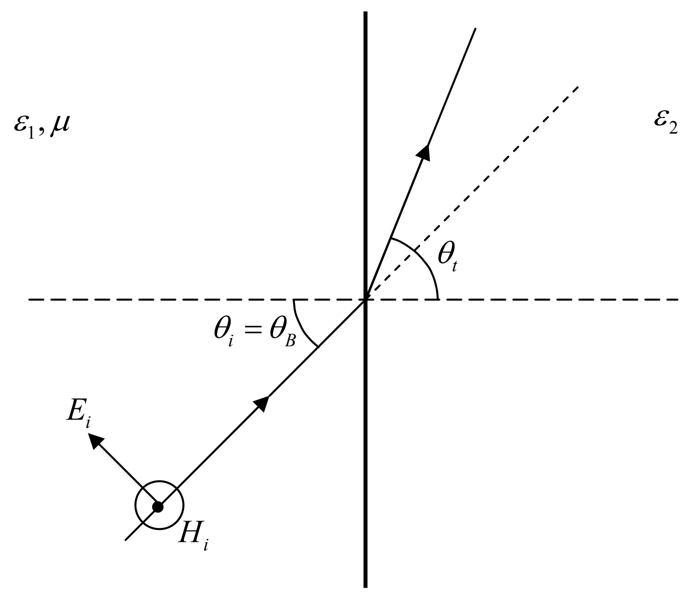
Problem 6.4

A TM wave is incident onto a medium with a dielectric permittivity ε_2 from a medium with dielectric permittivity ε_1 at the Brewster's angle of no reflection, θ_B. Both media have the same magnetic permeability $\mu_1 = \mu_2 \equiv \mu$. The reflection coefficient for a TM wave is

$$\frac{\hat{E}_r}{\hat{E}_i} = R = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

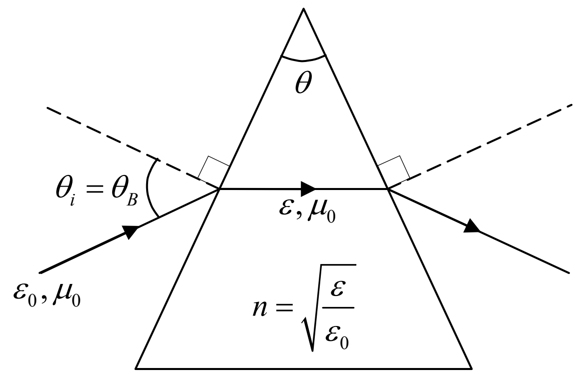
(a) What is the transmitted angle θ_t when $\theta_i = \theta_B$? How are θ_B and θ_t related?

(b) What is the Brewster angle of no reflection?



(c) What is the critical angle of transmission θ_C when $\mu_1 = \mu_2 \equiv \mu$? For the critical angle to exist, what must be the relationship between ε_1 and ε_2?

(d) A Brewster prism will pass TM polarized light without any loss from reflections.



For the light path through the prism shown above what is the apex angle θ? Evaluate for glass with $n = 1.45$.

(e) In the Brewster prism of part (d), determine the output power in terms of the incident power for TE polarized light with $n = 1.45$. The reflection coefficient for a TE wave is

$$\frac{\hat{E}_r}{\hat{E}_i} = R = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$