Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

FINAL EXAM, May 18, 2010

ANSWER BOOKLET

Your Full Name:	
Recitation Time :	o'clock

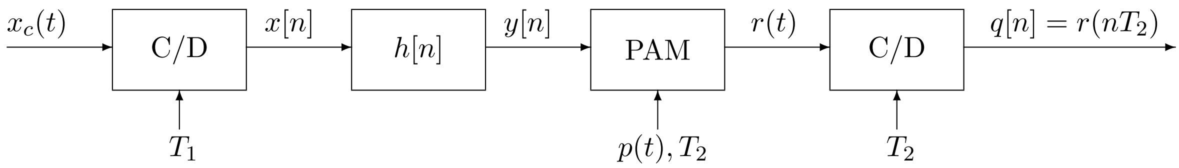
- This exam is **closed book**, but **4** sheets of notes are allowed. Calculators and other electronic aids will not be necessary and are not allowed.
- Check that this **ANSWER BOOKLET** has pages numbered up to 26. The booklet contains spaces for **all** relevant reasoning and answers.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to this booklet the work you would like us to look at. Let us know if you need additional scratch paper. **Only** this booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- There are **5 problems, weighted as shown, for a total of 100 points**. (The points indicated on the following pages for the various subparts of the problems are our best guess for now, but may be modified slightly when we get to grading.)

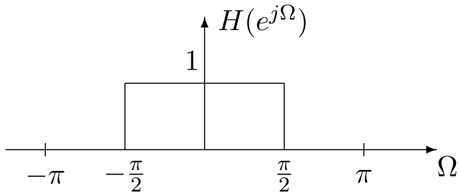
Problem	**Your Score**
1 (17 points)	
2 (18 points)	
3 (25 points)	
4 (20 points)	
5 (20 points)	
Total (100 points)	

Problem 1 (17 points)

Note that 1(d) does not depend on your answers to 1(a)-(c), and can be done independently of them.

$$X_c(j\omega) = 0 \quad \text{for} \quad |\omega| \geq 2\pi \times 10^3 \ .$$





1(a) (3 points) Determine the *largest* value of T_1 to ensure that $y[n] = x_c(nT_1)$.

Largest possible T_1 is:

1(b) (6 points) With T_1 picked as in 1(a), determine a choice for T_2 and $p(t)$ to ensure that

$$r(t) = x_c(t) \ .$$

(You can leave your expressions for T_2 and $p(t)$ in terms of T_1, instead of substituting in the numerical value you obtained in 1(a) for T_1.)

$T_2 =$ $p(t) =$

(**Continue 1(b) on next page**:)

1(b) (continued) Also determine if there is another choice of T_2 and $p(t)$ that could ensure the equality $r(t) = x_c(t)$. Explain your answer carefully.

Is there another possible choice of T_2? If you answer "Yes", then specify such an alternative T_2.

Is there another possible choice of $p(t)$? If you answer "Yes", then specify such an alternative $p(t)$.

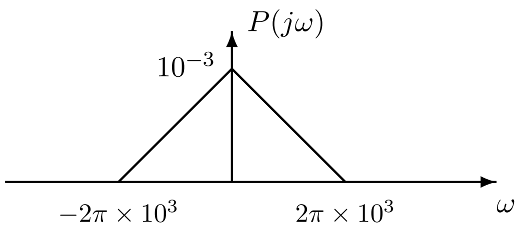
1(c) (3 points) With T_1 picked as in 1(a), how would you modify your choice of T_2 and $p(t)$ from 1(b) to ensure that

$$r(t) = x_c(2.7t) \ .$$

Modified $T_2 =$

Modified $p(t) =$

1(d) (4 points) Assume that $p(t)$ is now chosen so that its CTFT, $P(j\omega)$, is as shown below. Determine a value of T_2 to ensure that $q[n] = y[n]$.



An appropriate choice is $T_2 =$

Problem 2 (18 points)

For each of the following parts, write down whether the statement is **True** or **False** (circle whichever is appropriate), giving a clear explanation or counterexample. (Take care with this!)

2(a) (4 points) Suppose $x[n]$ is a zero-mean discrete-time (DT) wide-sense stationary (WSS) random process. If its autocorrelation function $R_{xx}[m]$ is 0 for $|m| \geq 2$ but nonzero for $m = -1, 0, 1$, then the linear minimum mean-square-error (LMMSE) estimator of $x[n+1]$ from measurements of $x[n]$ and $x[n-1]$, namely

$$\widehat{x}[n+1] = a_0 x[n] + a_1 x[n-1] \ ,$$

will necessarily have $a_1 = 0$.

TRUE **FALSE**

Explanation/counterexample:

2(b) (4 points) If the power spectral density $S_{yy}(j\omega)$ of a continuous-time (CT) WSS random process $y(t)$ is given by

$$S_{yy}(j\omega) = \frac{17 + \omega^2}{23 + \omega^2}$$

then the mean value of the process is zero, i.e., $\mu_y = E[y(t)] = 0$.

TRUE **FALSE**

Explanation/counterexample:

2(c) (4 points) If the autocovariance function $C_{vv}[m]$ of a DT WSS random process $v[n]$ is given by

$$C_{vv}[m] = \left(\frac{1}{3}\right)^{|m|},$$

then the LMMSE estimator of $v[n+1]$ from all past measurements, which we write as

$$\widehat{v}[n+1] = \left(\sum_{k=0}^{\infty} h_k v[n-k]\right) + d,$$

will have $h_k = 0$ for all $k \geq 1$, i.e., only the coefficients h_0 and d can be nonzero.

TRUE **FALSE**

Explanation/counterexample:

2(d) (3 points) The process $v[n]$ in 2(c) is ergodic in mean value.

TRUE **FALSE**

Explanation/counterexample:

2(e) (3 points) If $z[n] = v[n] + W$, where $v[n]$ is the process in 2(c), and where W is a random variable with mean 0 and variance $\sigma_W^2 > 0$, then the process $z[n]$ is ergodic in mean value.

TRUE **FALSE**

Explanation/counterexample:

Problem 3 (25 points)

$$\begin{aligned} \mathbf{q}[n+1] &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] + \mathbf{h}w[n] \ , \\ y[n] &= \mathbf{c}^T\mathbf{q}[n] + v[n] \ . \end{aligned}$$

where

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ 0 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

3(a) Determine the two natural frequencies of the system (i.e., the eigenvalues of **A**), and for each of them specify whether the associated mode satisfies the properties listed on the next page.

(**Write your answers on the next page.**)

3(a) (continued)(8 points)

The two eigenvalues are: $\lambda_1 =$ $\lambda_2 =$

List below whichever of the eigenvalues, if either, has an associated mode that satisfies the indicated condition:

(i) decays asymptotically to 0 in the zero-input response:

(ii) is reachable from the input $x[n]$ (with $w[n]$ kept at zero):

(iii) is reachable from the input $w[n]$ (with $x[n]$ kept at zero):

(iv) is observable from the output $y[n]$:

3(b) (2 points) Your specification of the observer, to obtain an estimate $\widehat{\mathbf{q}}[n]$ of the state $\mathbf{q}[n]$ (explain your choice):

3(c) (2 points) With $\widetilde{\mathbf{q}}[n] = \mathbf{q}[n] - \widehat{\mathbf{q}}[n]$, explain carefully why the components $\widetilde{q}_1[n]$ and $\widetilde{q}_2[n]$ of $\widetilde{\mathbf{q}}[n]$ at time n are uncorrelated with the noise terms $w[n]$ and $v[n]$ at time n (or — equivalently, of course! — explain why the components of $\widetilde{\mathbf{q}}[n+1]$ are uncorrelated with $w[n+1]$ and $v[n+1]$):

3(d) (4 points) The state estimation error in 3(c) is governed by a state-space model of the form

$$\widetilde{\mathbf{q}}[n+1] = \mathbf{B}\widetilde{\mathbf{q}}[n] + \mathbf{f}w[n] + \mathbf{g}v[n] \ .$$

Determine $\mathbf{B}$, $\mathbf{f}$ and $\mathbf{g}$ in terms of previously specified quantities.

$\mathbf{B} =$, $\mathbf{f} =$, $\mathbf{g} =$

3(e) (5 points) Is it possible to *arbitrarily* vary the natural frequencies of the state estimation error evolution equation in 3(d) by controlling the observer gains ℓ_1 and ℓ_2? Explicitly note how your answer here is consistent with your answer to 3(a)(iv).

What constraints, if any, on ℓ_1 and ℓ_2 must be satisfied to make the error evolution equation asymptotically stable?

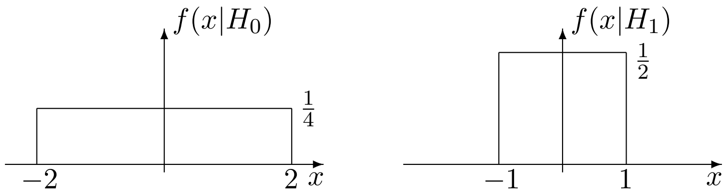
Would the choice $\ell_2 = 0$ allow you to obtain a good state estimate? — explain.

If you have done things correctly, you should find that choosing $\ell_1 = -\frac{3}{4}$ makes the matrix **B** in part 3(d) a diagonal matrix. **Keep ℓ_1 fixed at $-\frac{3}{4}$ for the rest of this problem**, and also assume ℓ_2 is chosen so that the error evolution equation is asymptotically stable.

3(f) (4 points) Under the given assumptions, the mean-squared estimation errors attain constant steady-state values, $E(\widetilde{q}_1^2[n]) = \sigma_{q1}^2$ and $E(\widetilde{q}_2^2[n]) = \sigma_{q2}^2$. Find explicit expressions for σ_{q1}^2 and σ_{q2}^2, expressing them as functions of ℓ_2. [**Hint**: At steady state, $E(\widetilde{q}_1^2[n+1]) = E(\widetilde{q}_1^2[n])$ and $E(\widetilde{q}_2^2[n+1]) = E(\widetilde{q}_2^2[n])$.]

$\sigma_{q1}^2 =$ $\sigma_{q2}^2 =$

Problem 4 (20 points)



4(a) (4 points) Sketch $\Lambda(x) = \frac{f_{X|H}(x|H_1)}{f_{X|H}(x|H_0)}$ as a function of x for $-2 < x < 2$:

4(b) (6 points)

(i) For threshold η at some value strictly above 2, determine P_D and P_{FA}:

$P_D =$ $P_{FA} =$

(ii) For η at some value strictly between 0 and 2, determine P_D and P_{FA}:

$P_D =$ $P_{FA} =$

4(b) (continued)

(iii) For η at some value strictly below 0, determine P_D and P_{FA}:

$P_D =$ $\qquad\qquad\qquad\qquad$ $P_{FA} =$

4(c) (2 points) If the specified limit on P_{FA} is $\beta = 0.3$, which of the choices in 4(b) can we pick, and what is the associated P_D?

4(d) (8 points) What is the probability that we get $\Lambda(X) = 0$ if H_0 holds? And what is the probability we get $\Lambda(X) = 0$ if H_1 holds?

$P(\Lambda(X) = 0 \,|\, H_0) =$

$P(\Lambda(X) = 0 \,|\, H_1) =$

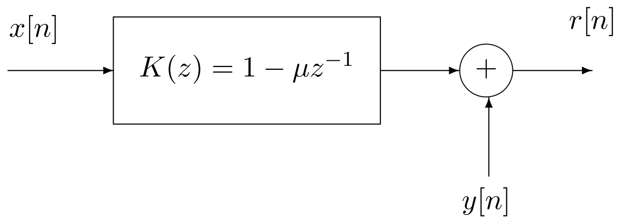
Announce 'H_0' when $\Lambda(x) = 0$. When $\Lambda(x) > 0$, announce 'H_1' with probability α, and otherwise announce 'H_0'. What are P_D and P_{FA} with this *randomized* decision rule?

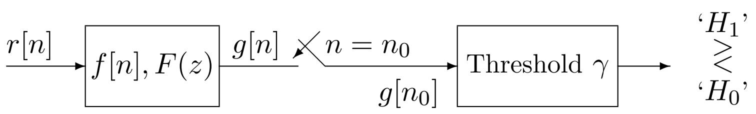
$P_D =$

$P_{FA} =$

To maximize P_D while keeping $P_{FA} \leq 0.3$ with this decision rule, choose $\alpha =$

Problem 5 (20 points)





5(a) (10 points) Suppose $x[n]$ is a signal that we are interested in, while $y[n]$ is a zero-mean, i.i.d., Gaussian noise process, with variance σ^2 at each instant of time.

$$\begin{aligned} H_0 : x[n] &= 0 , & P(H_0) &= p_0 , \\ H_1 : x[n] &= \delta[n] , & P(H_1) &= p_1 = 1 - p_0 . \end{aligned}$$

(i) Fully specify the MPE receiver when $n_0 = 0$, i.e., specify $f[n]$ or $F(z)$ and the value of γ.

(**Write your answers on the next page.**)

5(a)(i) (continued)

Specify $f[n]$ or $F(z)$:

Threshold $\gamma =$

5(a)(ii) Write down an expression for $P(\text{‘}H_1\text{’}|H_0)$ and for the minimum probability of error in the case where the two hypotheses are equally likely, $p_0 = p_1$. You can write these in terms of the standard function

$$Q(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{t^2}{2}} \, dt$$

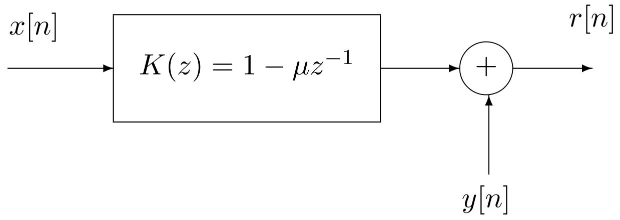
$P(\text{‘}H_1\text{’}|H_0) =$

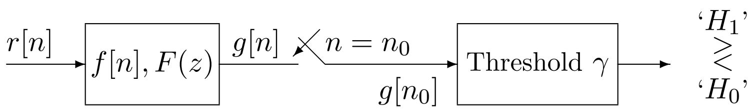
The minimum probability of error is:

5(a)(iii) If the value of μ is changed to a new value $\overline{\mu} = \mu/2$, we can get the same probability of error as prior to the change if the noise variance changes to some new value $\overline{\sigma}^2$. Express $\overline{\sigma}$ in terms of σ:

$\overline{\sigma} =$

5(b) (10 points) Suppose now that $x[n]$ is a zero-mean, i.i.d., Gaussian noise process, with variance σ^2 at each instant of time, and that $y[n]$ is the signal we are interested in. We





have the following two hypotheses regarding $y[n]$:

$$
\begin{aligned}
H_0 : y[n] &= 0 , & P(H_0) &= p_0 , \\
H_1 : y[n] &= \delta[n] , & P(H_1) &= p_1 = 1 - p_0 .
\end{aligned}
$$

Fully specify the MPE receiver when $n_0 = 0$, i.e., specify $f[n]$ or $F(z)$ and the value of γ for this case.

(Write your answers on the next page.)

5(b) (continued)

Specify $f[n]$ or $F(z)$:

Threshold $\gamma =$

Also write down (in terms of μ and σ) the relevant “signal energy to noise power” ratio that governs the performance of this system:

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