6.003: Signals and Systems

Fourier Transform

November 3, 2011

Last Time: Fourier Series

Representing periodic signals as sums of **sinusoids**.

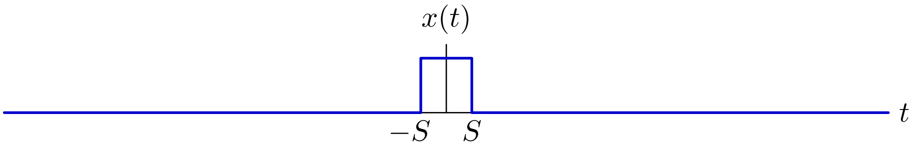
$\rightarrow$ new representations for systems as **filters**.

Today: generalize for **aperiodic** signals.

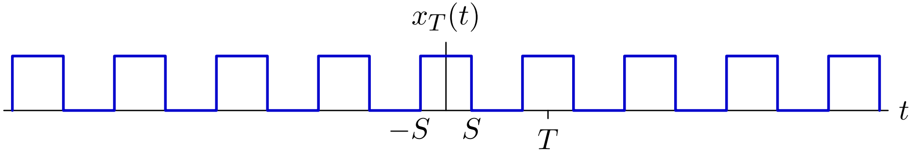
Fourier Transform

An aperiodic signal can be thought of as periodic with infinite period.

Let $x(t)$ represent an aperiodic signal.



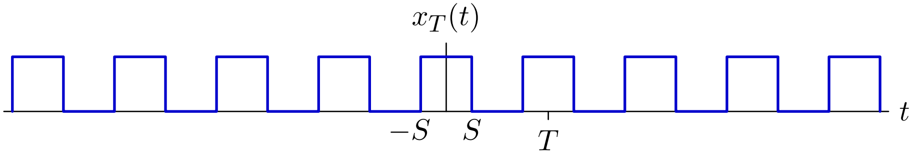
"Periodic extension": $x_T(t) = \sum_{k=-\infty}^{\infty} x(t + kT)$



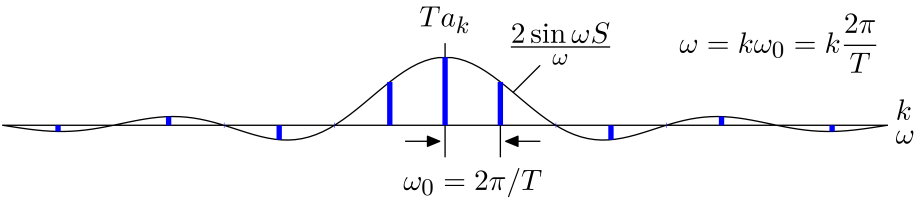
Then $x(t) = \lim_{T\to\infty} x_T(t)$.

Fourier Transform

Represent $x_T(t)$ by its Fourier series.

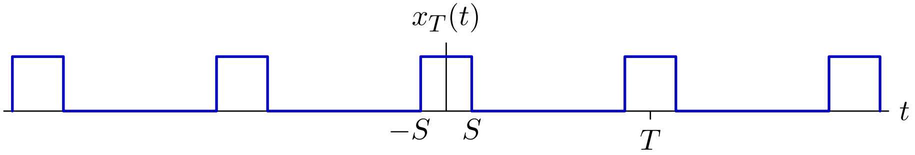


$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\,\frac{\sin\omega S}{\omega}$$

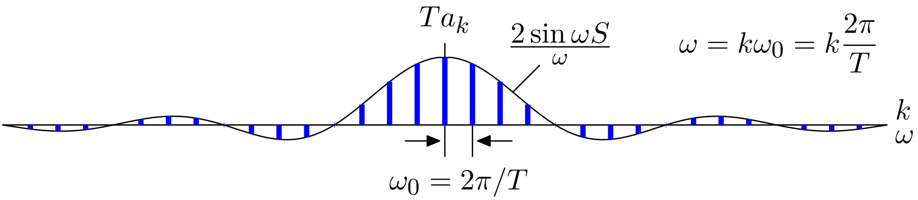


Fourier Transform

Doubling period doubles # of harmonics in given frequency interval.

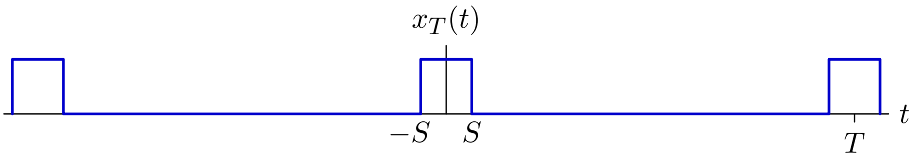


$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

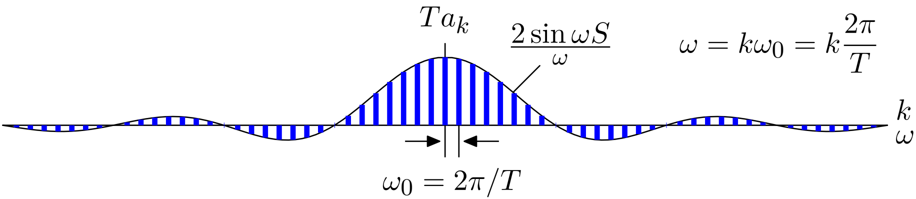


Fourier Transform

As $T \to \infty$, discrete harmonic amplitudes $\to$ a continuum $E(\omega)$.



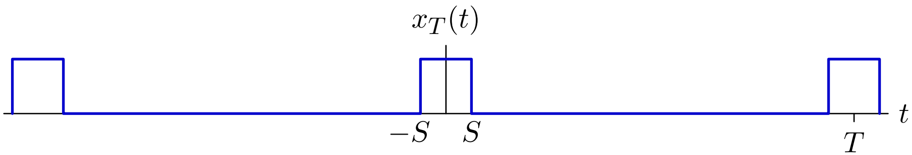
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

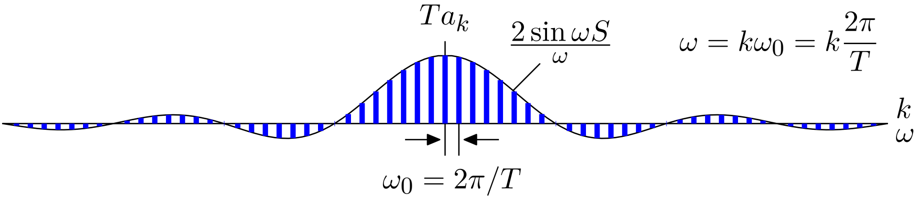


$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

Fourier Transform

As $T \to \infty$, synthesis sum $\to$ integral.





$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T}E(\omega)}_{a_k}\, e^{j\frac{2\pi}{T}kt} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi}E(\omega)e^{j\omega t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} E(\omega)e^{j\omega t}d\omega$$

Fourier Transform

Replacing $E(\omega)$ by $X(j\omega)$ yields the Fourier transform relations.

$$E(\omega) = X(j\omega)$$

Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{(“analysis” equation)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad \text{(“synthesis” equation)}$$

Form is similar to that of Fourier series
$\rightarrow$ provides alternate view of signal.

Relation between Fourier and Laplace Transforms

If the Laplace transform of a signal exists and if the ROC includes the $j\omega$ axis, then the Fourier transform is equal to the Laplace transform evaluated on the $j\omega$ axis.

Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Fourier transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(s)|_{s=j\omega}$$

Relation between Fourier and Laplace Transforms

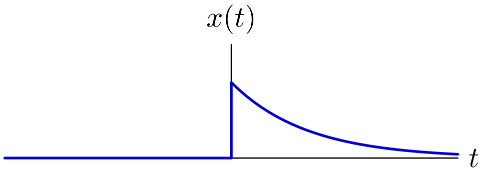
Fourier transform "inherits" properties of Laplace transform.

Property	$x(t)$	$X(s)$	$X(j\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$aX_1(j\omega) + bX_2(j\omega)$
Time shift	$x(t - t_0)$	$e^{-st_0}X(s)$	$e^{-j\omega t_0}X(j\omega)$
Time scale	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$	$\frac{1}{\lvert a \rvert}X\left(\frac{j\omega}{a}\right)$
Differentiation	$\frac{dx(t)}{dt}$	$sX(s)$	$j\omega X(j\omega)$
Multiply by t	$tx(t)$	$-\frac{d}{ds}X(s)$	$-\frac{1}{j}\frac{d}{d\omega}X(j\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(s) \times X_2(s)$	$X_1(j\omega) \times X_2(j\omega)$

Relation between Fourier and Laplace Transforms

There are also important differences.

Compare Fourier and Laplace transforms of $x(t) = e^{-t}u(t)$.



Laplace transform

$$X(s) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+1)t}dt = \frac{1}{1+s} \;;\; \mathrm{Re}(s) > -1$$

a complex-valued function of **complex** domain.

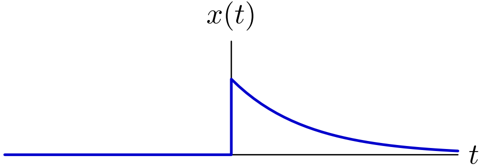
Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(j\omega+1)t}dt = \frac{1}{1+j\omega}$$

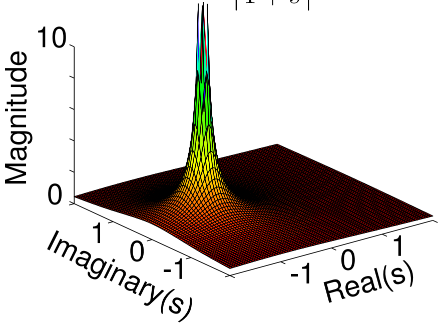
a complex-valued function of **real** domain.

Laplace Transform

The Laplace transform maps a function of time t to a complex-valued function of complex-valued domain s.

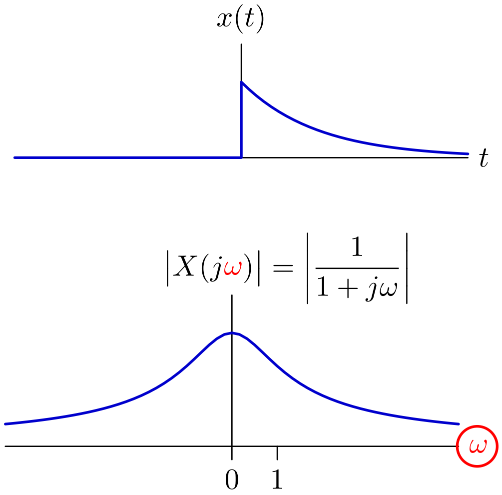


$$|X(s)| = \left|\frac{1}{1+s}\right|$$



Fourier Transform

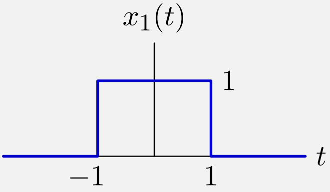
The Fourier transform maps a function of time t to a complex-valued function of real-valued domain ω.



Frequency plots provide intuition that is difficult to otherwise obtain.

Check Yourself

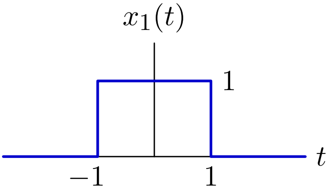
Find the Fourier transform of the following square pulse.



1. $X_1(j\omega) = \frac{1}{\omega}\left(e^{\omega} - e^{-\omega}\right)$
2. $X_1(j\omega) = \frac{1}{\omega}\sin\omega$
3. $X_1(j\omega) = \frac{2}{\omega}\left(e^{\omega} - e^{-\omega}\right)$
4. $X_1(j\omega) = \frac{2}{\omega}\sin\omega$
5. none of the above

Fourier Transform

Compare the Laplace and Fourier transforms of a square pulse.



Laplace transform:

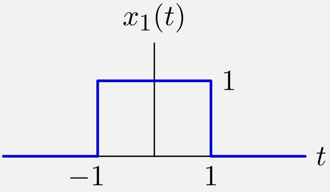
$$X_1(s) = \int_{-1}^{1} e^{-st} dt = \left.\frac{e^{-st}}{-s}\right|_{-1}^{1} = \frac{1}{s}\ \ e^s - e^{-s} \qquad \text{[function of } s = \sigma + j\omega\text{]}$$

Fourier transform

$$X_1(j\omega) = \int_{-1}^{1} e^{-j\omega t} dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1}^{1} = \frac{2\sin\omega}{\omega} \qquad \text{[function of } \omega\text{]}$$

Check Yourself

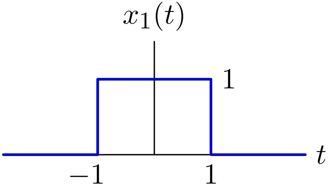
Find the Fourier transform of the following square pulse. 4



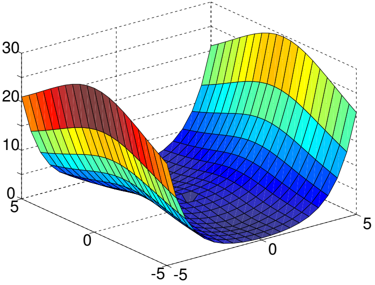
1. $X_1(j\omega) = \frac{1}{\omega}\left(e^{\omega} - e^{-\omega}\right)$
2. $X_1(j\omega) = \frac{1}{\omega}\sin\omega$
3. $X_1(j\omega) = \frac{2}{\omega}\left(e^{\omega} - e^{-\omega}\right)$
4. $X_1(j\omega) = \frac{2}{\omega}\sin\omega$
5. none of the above

Laplace Transform

Laplace transform: complex-valued function of complex domain.



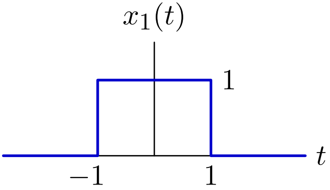
$$|X(s)| = \left| \frac{1}{s}(e^s - e^{-s}) \right|$$



Fourier Transform

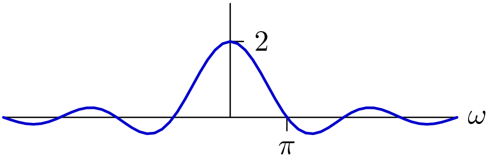
The Fourier transform is a function of real domain: frequency ω.

Time representation:



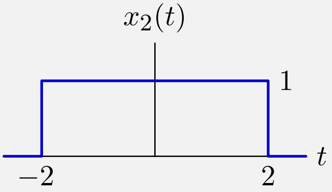
Frequency representation:

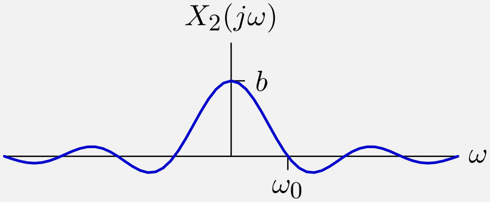
$$X_1(j\omega) = \frac{2\sin\omega}{\omega}$$



Check Yourself

Signal $x_2(t)$ and its Fourier transform $X_2(j\omega)$ are shown below.





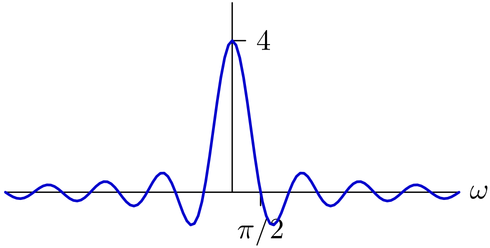
Which is true?

1. $b = 2$ and $\omega_0 = \pi/2$
2. $b = 2$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above

Check Yourself

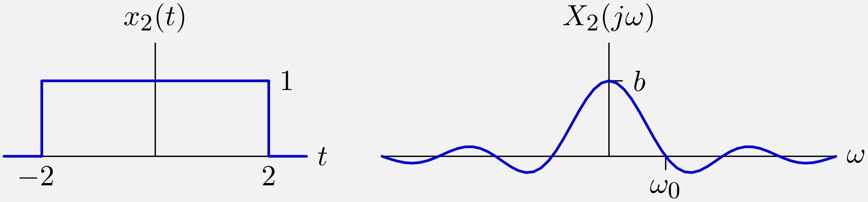
Find the Fourier transform.

$$X_2(j\omega) = \int_{-2}^{2} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-2}^{2} = \frac{2\sin 2\omega}{\omega} = \frac{4\sin 2\omega}{2\omega}$$



Check Yourself

Signal $x_2(t)$ and its Fourier transform $X_2(j\omega)$ are shown below.

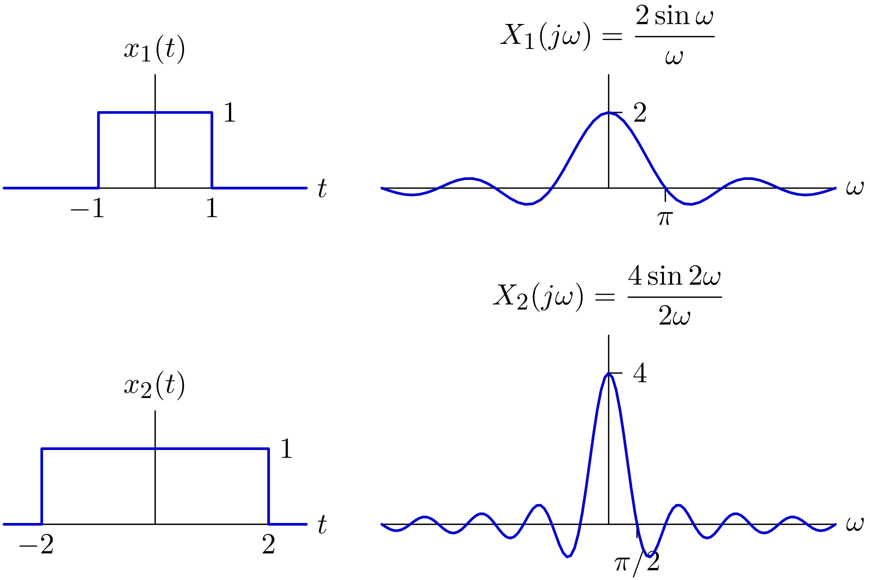


Which is true? 3

1. $b = 2$ and $\omega_0 = \pi/2$
2. $b = 2$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above

Fourier Transforms

Stretching time compresses frequency.



Check Yourself

Stretching time compresses frequency.

Find a general scaling rule.

Let $x_2(t) = x_1(at)$.

If time is stretched in going from x_1 to x_2, is $a > 1$ or $a < 1$?

Check Yourself

Stretching time compresses frequency.

Find a general scaling rule.

Let $x_2(t) = x_1(at)$.

If time is stretched in going from x_1 to x_2, is $a > 1$ or $a < 1$?

$$x_2(2) = x_1(1)$$

$$x_2(t) = x_1(at)$$

Therefore $a = 1/2$, or more generally, $a < 1$.

Check Yourself

Stretching time compresses frequency.

Find a general scaling rule.

Let $x_2(t) = x_1(at)$.

If time is stretched in going from x_1 to x_2, is $a > 1$ or $a < 1$?

$a < 1$

Fourier Transforms

Find a general scaling rule.

Let $x_2(t) = x_1(at)$.

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x_1(at)e^{-j\omega t}dt$$

Let $\tau = at$ $(a > 0)$.

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau/a}\ \frac{1}{a}\,d\tau = \frac{1}{a}X_1\left(\frac{j\omega}{a}\right)$$

If $a < 0$ the sign of $d\tau$ would change along with the limits of integration. In general,

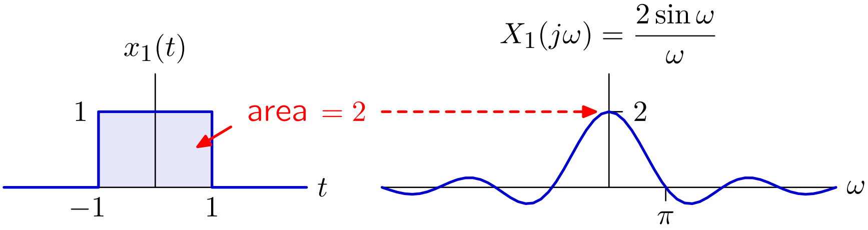
$$x_1(at) \ \leftrightarrow\ \frac{1}{|a|}X_1\left(\frac{j\omega}{a}\right).$$

If time is stretched ($a < 1$) then frequency is compressed and amplitude increases (preserving area).

Moments

The value of $X(j\omega)$ at $\omega = 0$ is the integral of $x(t)$ over time t.

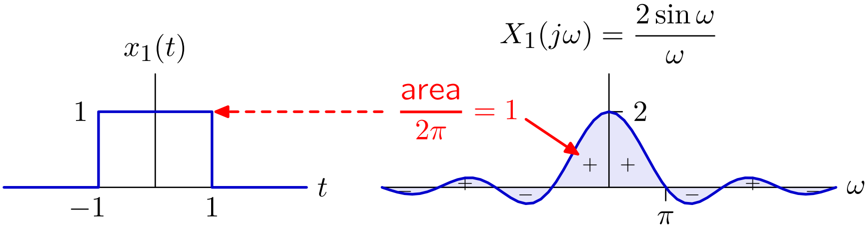
$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j0t}dt = \int_{-\infty}^{\infty} x(t)\,dt$$



Moments

The value of $x(0)$ is the integral of $X(j\omega)$ divided by 2π.

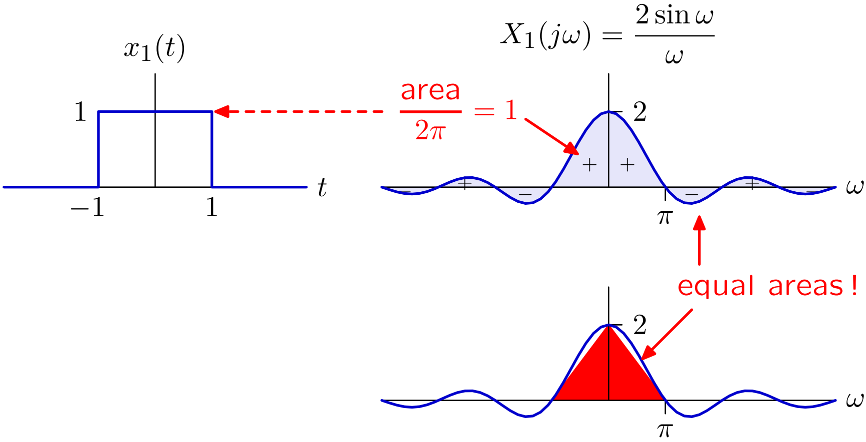
$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega$$



Moments

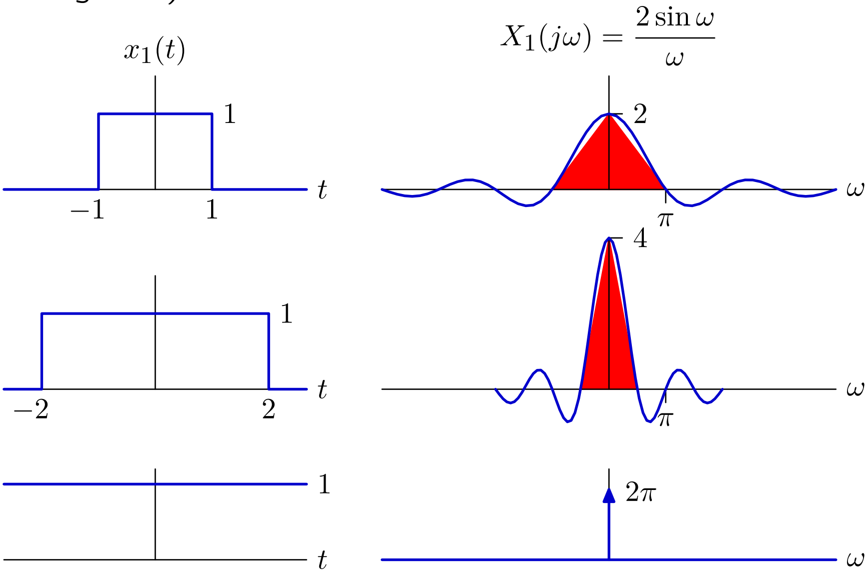
The value of $x(0)$ is the integral of $X(j\omega)$ divided by 2π.

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega$$



Stretching to the Limit

Stretching time compresses frequency and increases amplitude (preserving area).



New way to think about an impulse!

Fourier Transform

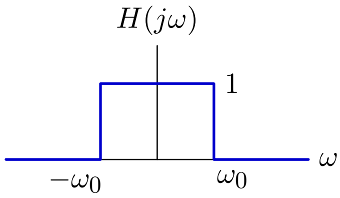
One of the most useful features of the Fourier transform (and Fourier series) is the simple "inverse" Fourier transform.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{(Fourier transform)}$$

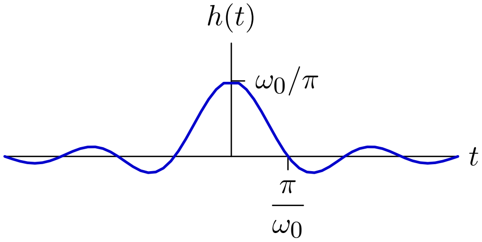
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad \text{("inverse" Fourier transform)}$$

Inverse Fourier Transform

Find the impulse reponse of an “ideal” low pass filter.



$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega t}d\omega = \frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_0}^{\omega_0} = \frac{\sin\omega_0 t}{\pi t}$$



This result is not so easily obtained without inverse relation.

Fourier Transform

The Fourier transform and its inverse have very similar forms.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{(Fourier transform)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{(“inverse” Fourier transform)}$$

Convert one to the other by

- $t \to \omega$
- $\omega \to -t$
- scale by 2π

Duality

The Fourier transform and its inverse have very similar forms.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Two differences:

- minus sign: flips time axis (or equivalently, frequency axis)
- divide by 2π (or multiply in the other direction)

$$x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega)$$

$\omega \rightarrow t$ $\qquad$ $t \rightarrow \omega$; flip ; $\times 2\pi$

$$x_2(t) = g(t) \leftrightarrow X_2(j\omega) = 2\pi f(-\omega)$$

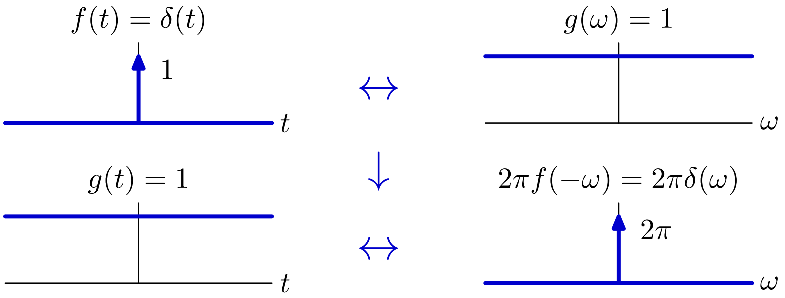
Duality

Using duality to find new transform pairs.

$$x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega)$$

$\omega \to t$ $\qquad$ $t \to \omega$; flip ; $\times 2\pi$

$$x_2(t) = g(t) \leftrightarrow X_2(j\omega) = 2\pi f(-\omega)$$

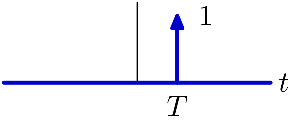


The function $g(t) = 1$ does not have a Laplace transform!

More Impulses

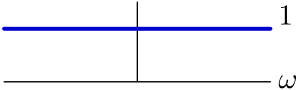
Fourier transform of delayed impulse: $\delta(t-T) \leftrightarrow e^{-j\omega T}$.

$$x(t) = \delta(t-T)$$

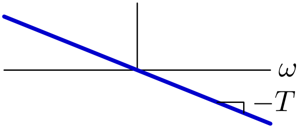


$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t-T)e^{-j\omega t}dt = e^{-j\omega T}$$

$$\left|X(j\omega)\right| = 1$$



$$\angle X(j\omega) = -\omega T$$



Eternal Sinusoids

Using duality to find the Fourier transform of an eternal sinusoid.

$$\delta(t - T) \quad \leftrightarrow \quad e^{-j\omega T}$$

$$\omega \rightarrow t \qquad\qquad t \rightarrow \omega \; ; \quad \text{flip} \; ; \quad \times 2\pi$$

$$e^{-jtT} \quad \leftrightarrow \quad 2\pi\delta(\omega + T)$$

$$T \rightarrow \omega_0 :$$

$$e^{-j\omega_0 t} \quad \leftrightarrow \quad 2\pi\delta(\omega + \omega_0)$$

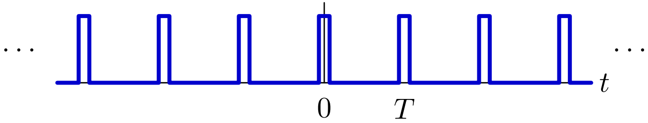
$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFS}}{\longleftrightarrow} \quad \{a_k\}$$

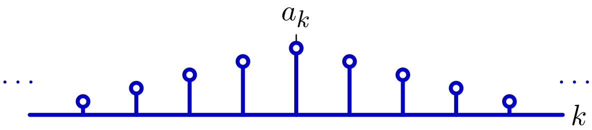
$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi a_k \delta \quad \omega - \frac{2\pi}{T}k$$

Relation between Fourier Transform and Fourier Series

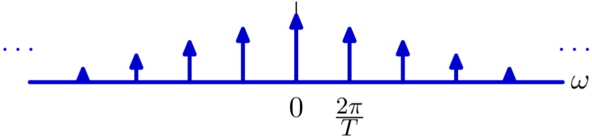
Each term in the Fourier series is replaced by an impulse.

$$x(t) = \sum_{k=-\infty}^{\infty} x_p(t - kT)$$





$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\frac{2\pi}{T})$$



Summary

Fourier transform generalizes ideas from Fourier series to aperiodic signals.

Fourier transform is strikingly similar to Laplace transform

- similar properties (linearity, differentiation, ...)
- but has a **simple inverse** (great for computation!)

Next time – applications (demos) of Fourier transforms