# 6.003: Signals and Systems

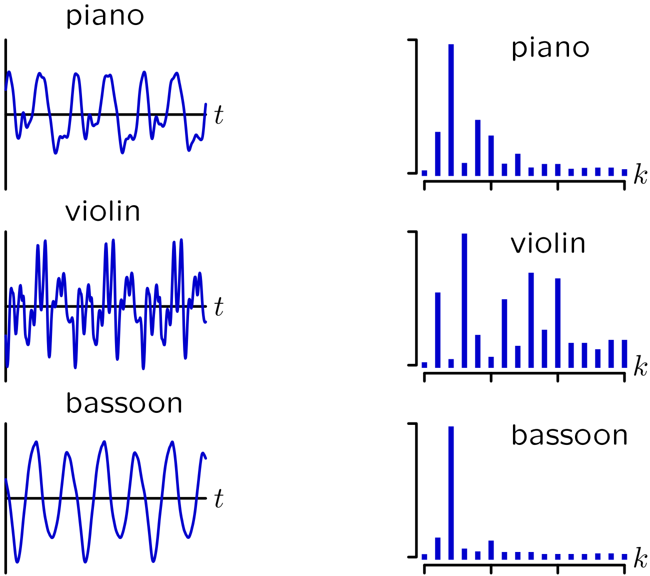
## Fourier Series

*November 1, 2011*

# Last Time: Describing Signals by Frequency Content

Harmonic content is natural way to describe some kinds of signals.

Ex: musical instruments (http://theremin.music.uiowa.edu/MIS.html)



# Last Time: Fourier Series

Determining harmonic components of a periodic signal.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$ (“analysis” equation)

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$ (“synthesis” equation)

We can think of Fourier series as an **orthogonal decomposition**.

# Orthogonal Decompositions

**Vector representation of 3-space:** let $\bar{r}$ represent a vector with components $\{x$, $y$, and $z\}$ in the $\{\hat{x}$, $\hat{y}$, and $\hat{z}\}$ directions, respectively.

$$x = \bar{r} \cdot \hat{x}$$
$$y = \bar{r} \cdot \hat{y}$$
$$z = \bar{r} \cdot \hat{z}$$

("analysis" equations)

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

("synthesis" equation)

**Fourier series:** let $x(t)$ represent a signal with harmonic components $\{a_0, a_1, \ldots, a_k\}$ for harmonics $\{e^{j0t}, e^{j\frac{2\pi}{T}t}, \ldots, e^{j\frac{2\pi}{T}kt}\}$ respectively.

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# Orthogonal Decompositions

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# Orthogonal Decompositions

Integrating over a period **sifts** out the $k^{\text{th}}$ component of the series.

Sifting as a dot product:

$$x = \bar{r} \cdot \hat{x} \equiv |\bar{r}||\hat{x}| \cos\theta$$

Sifting as an inner product:

$$a_k = e^{j\frac{2\pi}{T}kt} \cdot x(t) \equiv \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi}{T}kt} dt$$

where

$$a(t) \cdot b(t) = \frac{1}{T}\int_T a^*(t) b(t) dt\,.$$

The complex conjugate ($^*$) makes the inner product of the $k^{\text{th}}$ and $m^{\text{th}}$ components equal to $1$ iff $k = m$:

$$\frac{1}{T}\int_T \left(e^{j\frac{2\pi}{T}kt}\right)^* \left(e^{j\frac{2\pi}{T}mt}\right) dt = \frac{1}{T}\int_T e^{-j\frac{2\pi}{T}kt} e^{j\frac{2\pi}{T}mt} dt = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$

# Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$?

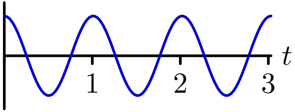
1. $\cos 2\pi t \perp \sin 2\pi t$ ?
2. $\cos 2\pi t \perp \cos 4\pi t$ ?
3. $\cos 2\pi t \perp \sin \pi t$ ?
4. $\cos 2\pi t \perp e^{j2\pi t}$ ?

# Check Yourself

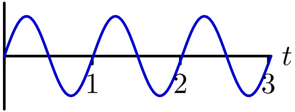
How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \sin 2\pi t$ ?

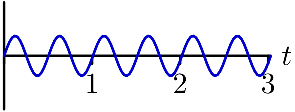
$\cos 2\pi t$




$\sin 2\pi t$




$\cos 2\pi t \sin 2\pi t = \frac{1}{2} \sin 4\pi t$




$\int_0^3 dt = 0$ therefore YES

# Check Yourself

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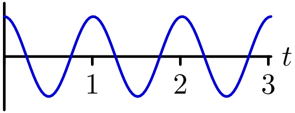
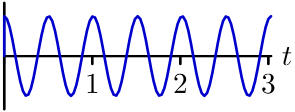
$\cos 2\pi t \perp \cos 4\pi t$ ?

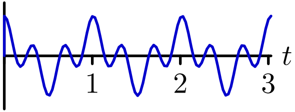
$\cos 2\pi t$




$\cos 4\pi t$




$$\cos 2\pi t \cos 4\pi t = \tfrac{1}{2}\cos 6\pi t + \tfrac{1}{2}\cos 2\pi t$$

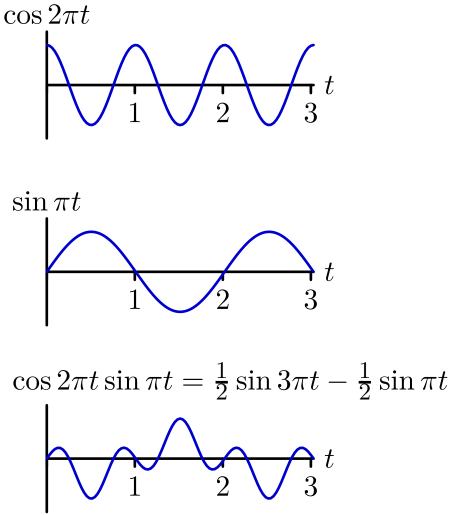



$\int_0^3 dt = 0$ therefore YES

# Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \sin \pi t$ ?




$\int_0^3 dt \neq 0$ therefore NO

# Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp e^{2\pi t}$ ?

$$e^{2\pi t} = \cos 2\pi t + j \sin 2\pi t$$

$\cos 2\pi t \perp \sin 2\pi t$ but not $\cos 2\pi t$

Therefore NO

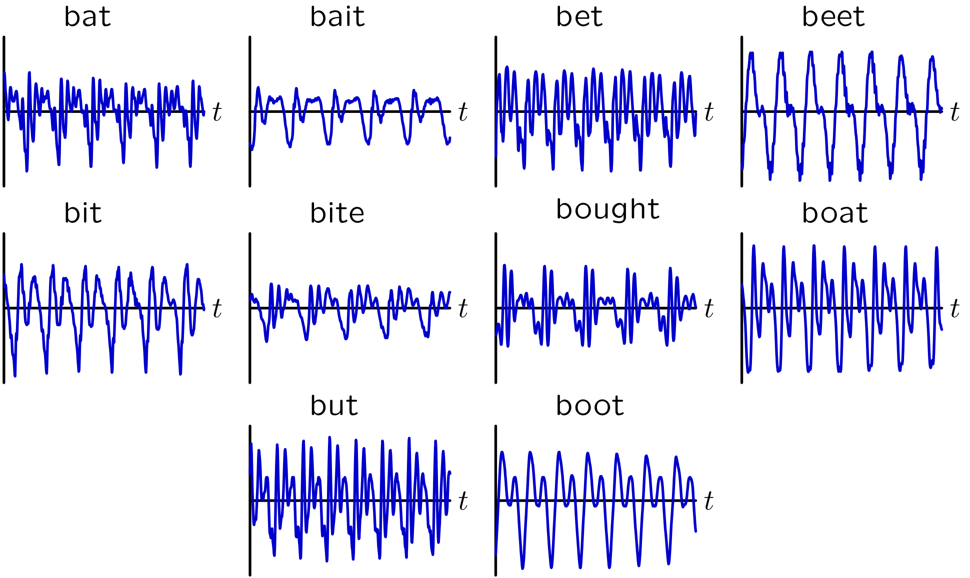
# Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$? 2

1. $\cos 2\pi t \perp \sin 2\pi t$ ? √
2. $\cos 2\pi t \perp \cos 4\pi t$ ? √
3. $\cos 2\pi t \perp \sin \pi t$ ? X
4. $\cos 2\pi t \perp e^{j2\pi t}$ ? X

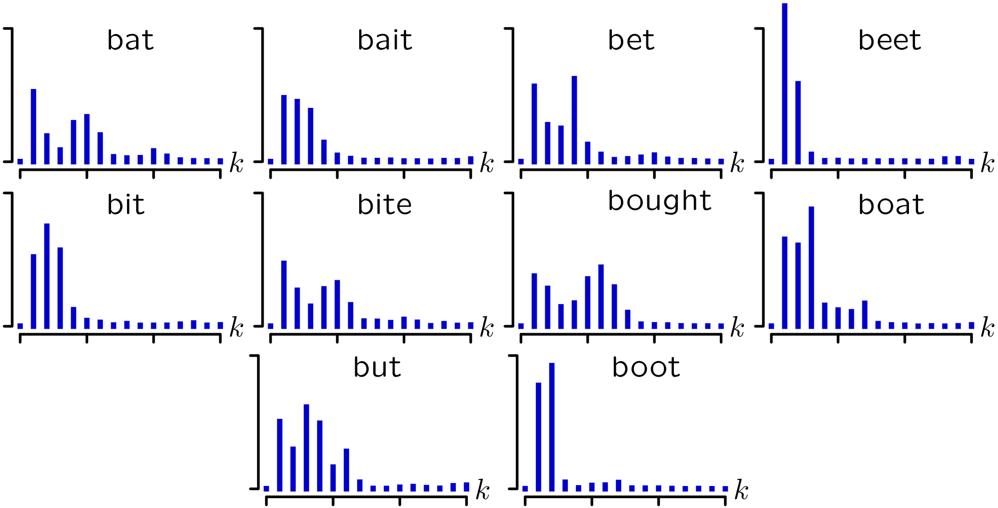
# Speech

Vowel sounds are quasi-periodic.



# Speech

Harmonic content is natural way to describe vowel sounds.



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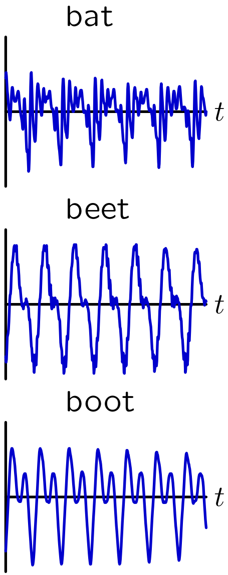
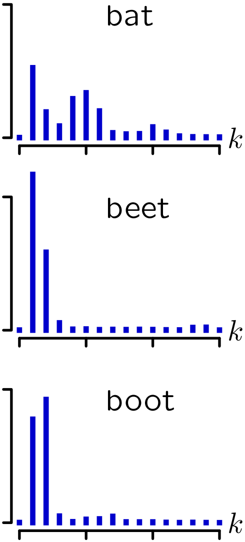
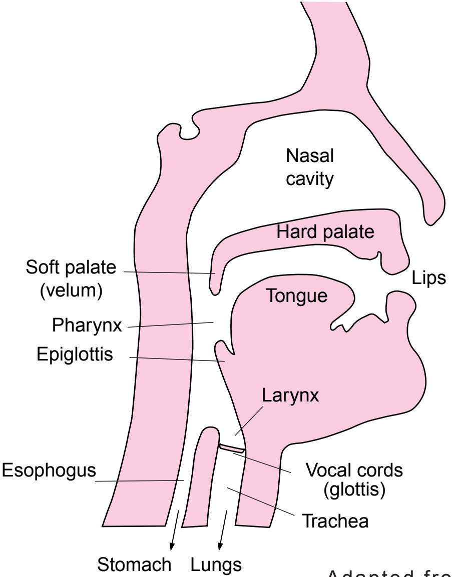


# Speech Production

Speech is generated by the passage of air from the lungs, through the vocal cords, mouth, and nasal cavity.

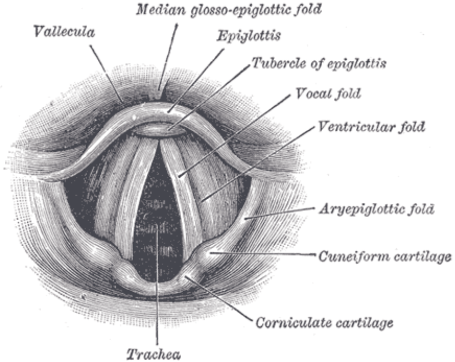



Adapted from T.F. Weiss

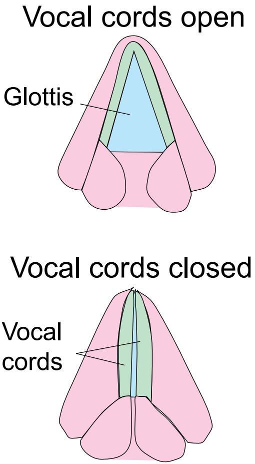
# Speech Production

Controlled by complicated muscles, vocal cords are set in vibration by the passage of air from the lungs.

Looking down the throat:



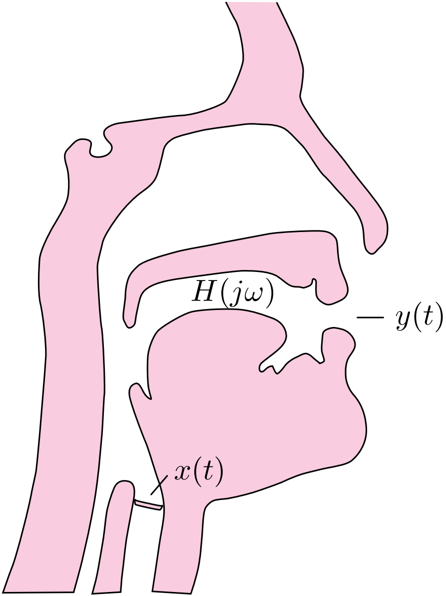

Gray's Anatomy




Adapted from T.F. Weiss

# Speech Production

Vibrations of the vocal cords are “filtered” by the mouth and nasal cavities to generate speech.



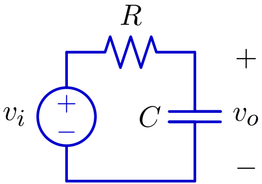
# Filtering

Notion of a filter.

LTI systems

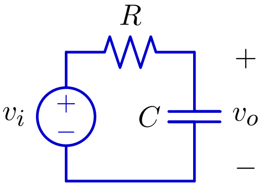
- cannot create new frequencies.
- can only scale magnitudes & shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit

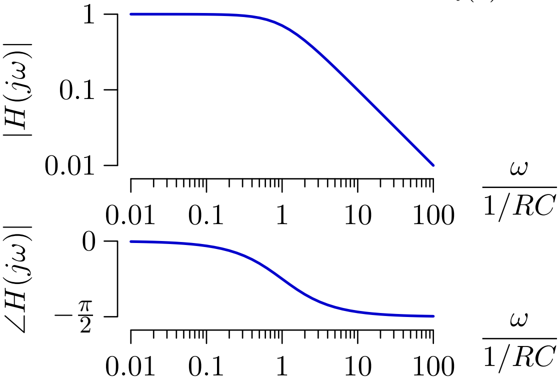


# Lowpass Filter

Calculate the frequency response of an RC circuit.

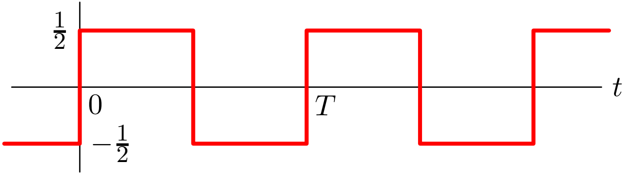



$$
\begin{aligned}
\text{KVL:} \quad & v_i(t) &&= Ri(t) + v_o(t) \\
\text{C:} \quad & i(t) &&= C\dot{v}_o(t) \\
\text{Solving:} \quad & v_i(t) &&= RC\dot{v}_o(t) + v_o(t) \\
& V_i(s) &&= (1 + sRC)V_o(s) \\
& H(s) &&= \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}
\end{aligned}
$$

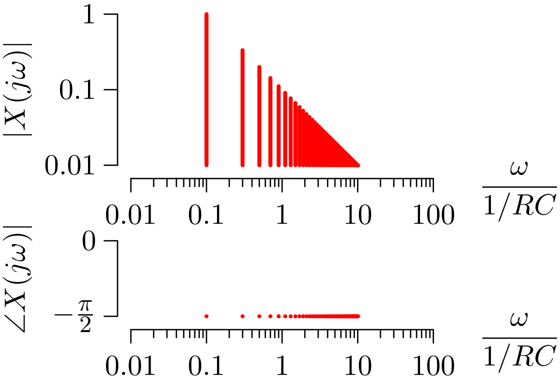


# Lowpass Filtering

Let the input be a square wave.

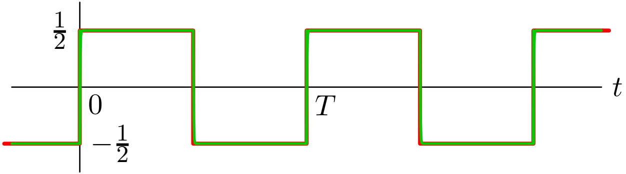



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \; ; \qquad \omega_0 = \frac{2\pi}{T}$$

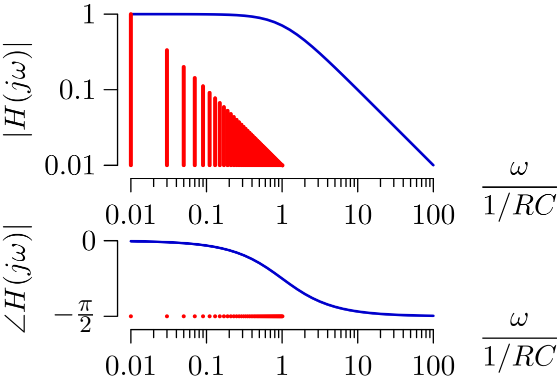


# Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

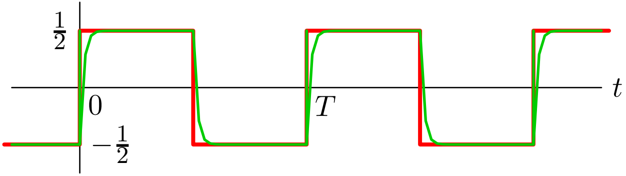



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

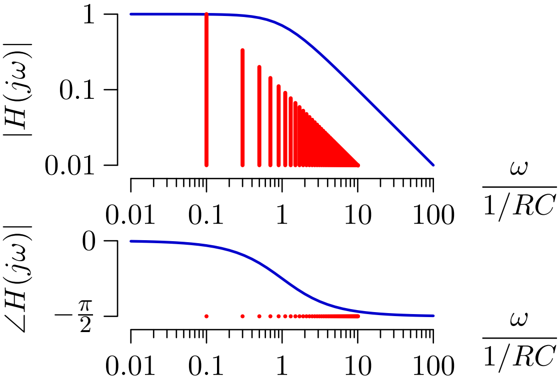


# Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

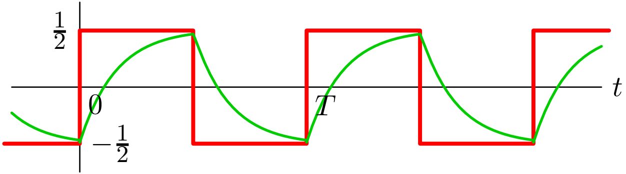



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \quad \omega_0 = \frac{2\pi}{T}$$

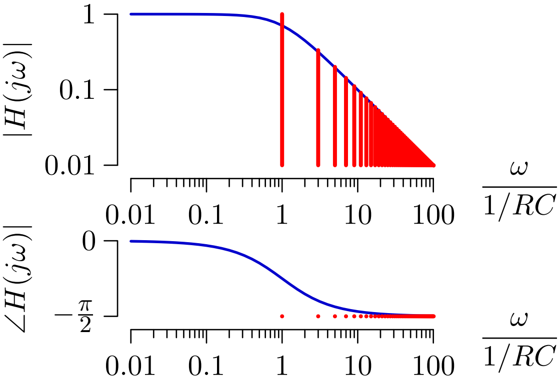


# Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.

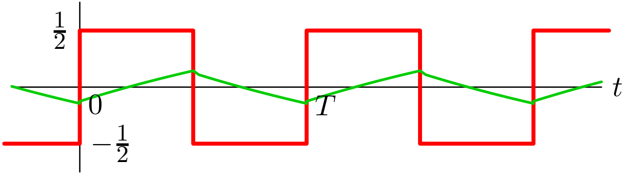



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



# Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.

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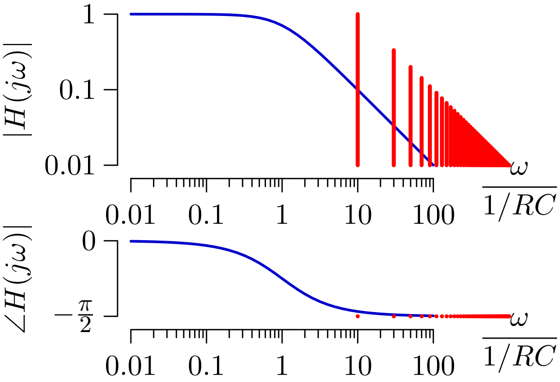
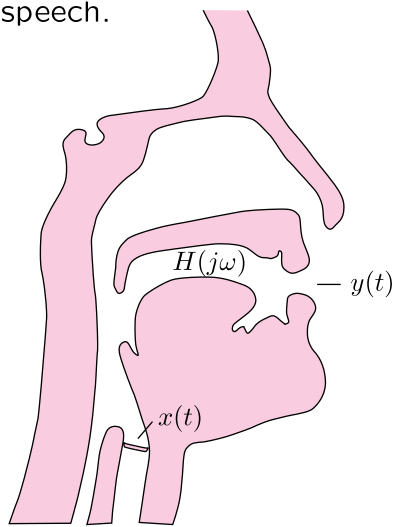
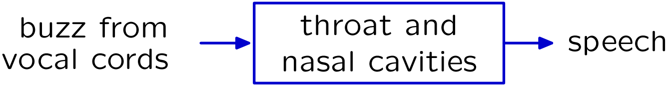


# Source-Filter Model of Speech Production

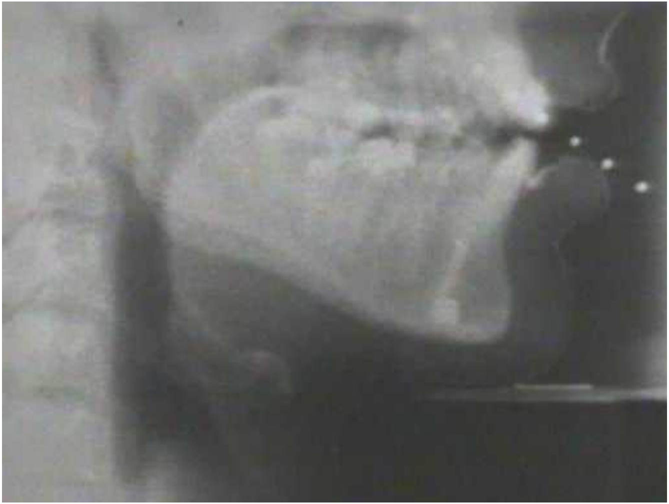
Vibrations of the vocal cords are "filtered" by the mouth and nasal cavities to generate speech.



# Speech Production

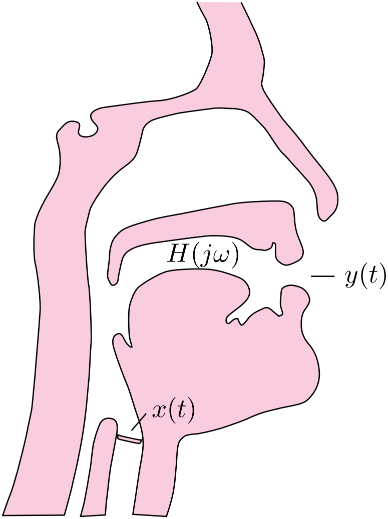
X-ray movie showing speech in production.

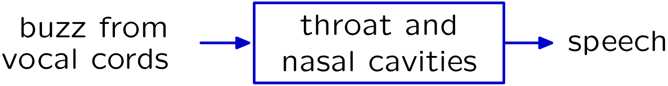


Courtesy of Kenneth N. Stevens. Used with permission.

# Demonstration

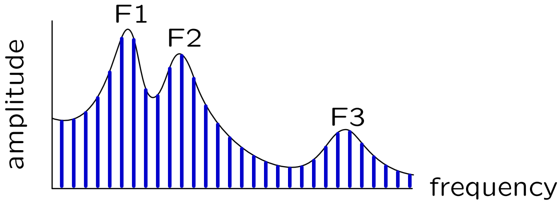
Artificial speech.



# Formants

Resonant frequencies of the vocal tract.

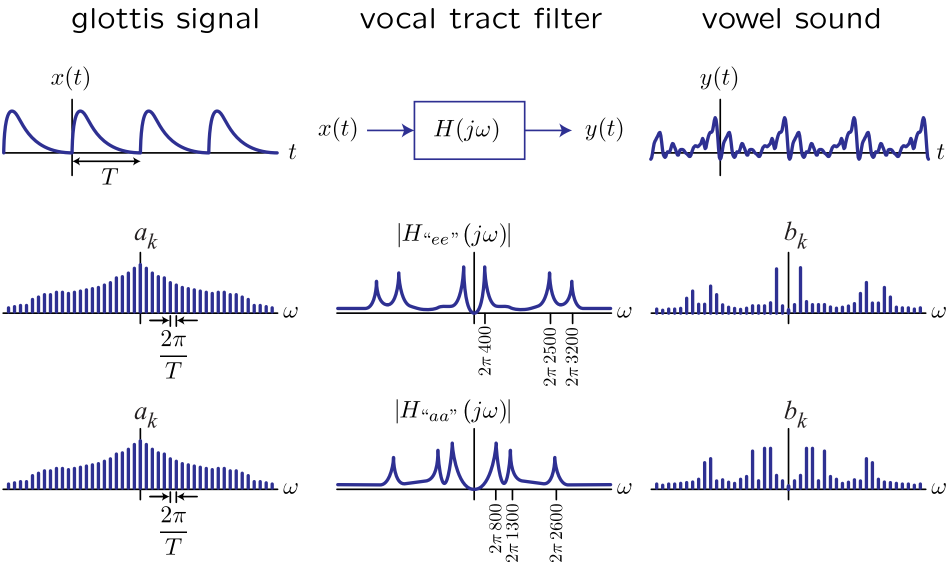



| | Formant | heed | head | had | hod | haw'd | who'd |
|---|---|---|---|---|---|---|---|
| Men | F1 | 270 | 530 | 660 | 730 | 570 | 300 |
| | F2 | 2290 | 1840 | 1720 | 1090 | 840 | 870 |
| | F3 | 3010 | 2480 | 2410 | 2440 | 2410 | 2240 |
| Women | F1 | 310 | 610 | 860 | 850 | 590 | 370 |
| | F2 | 2790 | 2330 | 2050 | 1220 | 920 | 950 |
| | F3 | 3310 | 2990 | 2850 | 2810 | 2710 | 2670 |
| Children | F1 | 370 | 690 | 1010 | 1030 | 680 | 430 |
| | F2 | 3200 | 2610 | 2320 | 1370 | 1060 | 1170 |
| | F3 | 3730 | 3570 | 3320 | 3170 | 3180 | 3260 |

http://www.sfu.ca/sonic-studio/handbook/Formant.html

# Speech Production

Same glottis signal + different formants → different vowels.




We detect changes in the filter function to recognize vowels.

# Singing

We detect changes in the filter function to recognize vowels … at least sometimes.

Demonstration.

"la" scale.

"lore" scale.

"loo" scale.

"ler" scale.

"lee" scale.

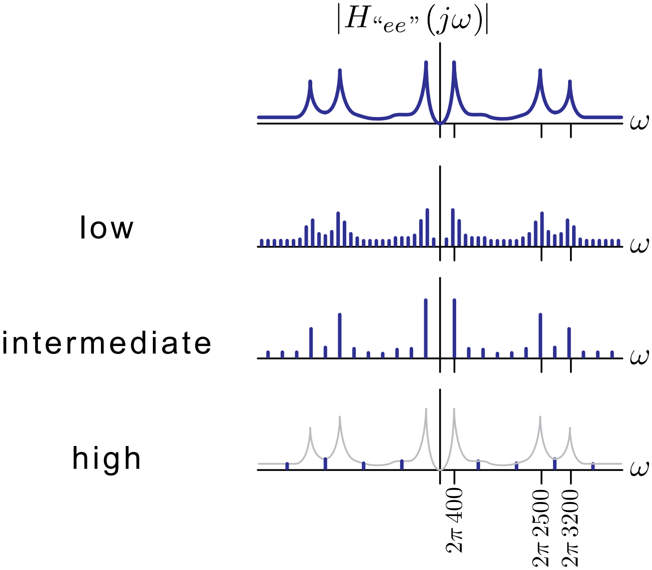
Low Frequency: "la" "lore" "loo" "ler" "lee".

High Frequency: "la" "lore" "loo" "ler" "lee".

http://www.phys.unsw.edu.au/jw/soprane.html

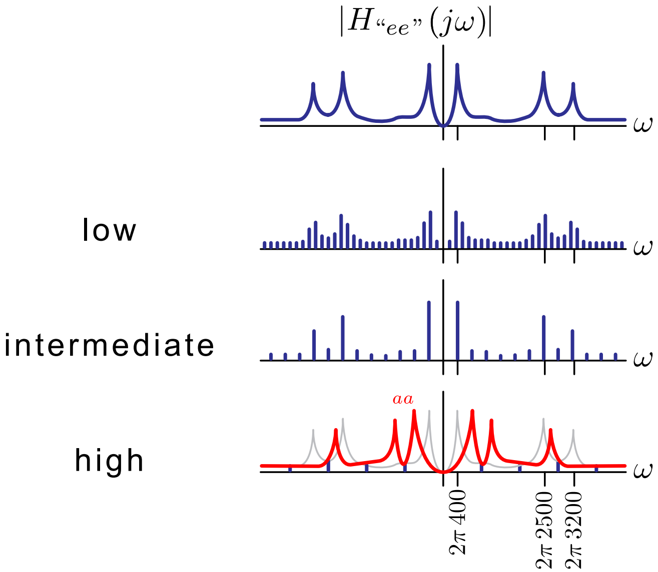
# Speech Production

We detect changes in the filter function to recognize vowels.



# Speech Production

We detect changes in the filter function to recognize vowels.



# Continuous-Time Fourier Series: Summary

Fourier series represent signals by their frequency content.

Representing a signal by its frequency content is useful for many signals, e.g., music.

Fourier series motivate a new representation of a system as a filter.

Representing a system as a filter is useful for many systems, e.g., speech synthesis.