6.003 (Fall 2009)

Quiz #3 *November 18, 2009*

Name:

Kerberos Username:

Please circle your section number:

Section	*Instructor*	*Time*
1	Marc Baldo	10 am
2	Marc Baldo	11 am
3	Elfar Adalsteinsson	1 pm
4	Elfar Adalsteinsson	2 pm

Partial credit will be given for answers that demonstrate some but not all of the important conceptual issues.

Explanations are not required and will not affect your grade.

You have **two hours.**

Please put your initials on all subsequent sheets.

Enter your answers in the boxes.

This quiz is closed book, but you may use three 8.5×11 sheets of paper (six sides total).

No calculators, computers, cell phones, music players, or other aids.

1	/12
2	/20
3	/18
4	/25
5	/25
Total	/100

1. Impulsive Input *[12 points]*

Let the following periodic signal

$$x(t) = \sum_{m=-\infty}^{\infty} \delta(t-3m) + \delta(t-1-3m) - \delta(t-2-3m)$$

be the input to an LTI system with system function

$$H(s) = e^{s/4} - e^{-s/4}\,.$$

Let b_k represent the Fourier series coefficients of the resulting output signal $y(t)$. Determine b_3.

$b_3 =$ $\boxed{j\frac{2}{3}}$

The period of $x(t)$ is $T = 3$. Therefore the period of $y(t)$ is also $T = 3$. The fundamental frequency of $x(t)$ (and $y(t)$) is $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$.

Let a_k represent the Fourier series coefficients for $x(t)$. Then

$$a_k = \frac{1}{3}\int_0^3 \left(\delta(t) + \delta(t-1) - \delta(t-2)\right) e^{-j\frac{2\pi}{3}kt}dt = \frac{1}{3}\left(1 + e^{-j\frac{2\pi}{3}k} - e^{-j\frac{2\pi}{3}k2}\right)$$

$$a_3 = \frac{1}{3}$$

The frequency response of the system is given by

$$H(j\omega) = e^{j\omega/4} - e^{-j\omega/4} = j2\sin\frac{\omega}{4}$$

Therefore

$$b_k = H\left(j\frac{2\pi}{3}k\right)a_k = \left(j2\sin\frac{2\pi k}{12}\right)a_k$$

and

$$b_3 = \left(j2\sin\frac{\pi}{2}\right)\frac{1}{3} = j\frac{2}{3}\,.$$

2. System Design *[20 points]*

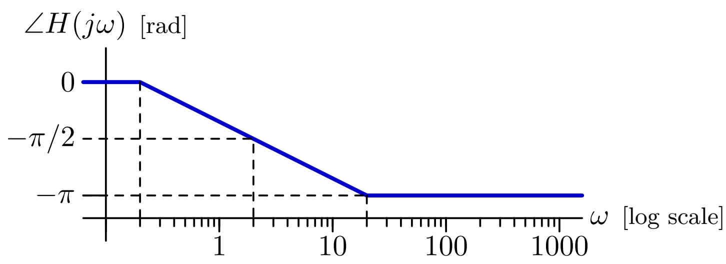
Design a stable CT LTI system H with **all** of the following three properties:

- the impulse response $h(t)$ has the form

$$h(t) = C\delta(t) + De^{-2t}u(t)$$

where C and D are real-valued constants,

- the angle of $H(j\omega)$ has the following straight-line approximation



- if the input $x(t)$ is 1 for all time, then the output $y(t)$ is 1 for all time.

Determine the system function $H(s)$ that is consistent with these design specifications. If no such a system exists, enter **none**.

$H(s) =$ $\frac{2-s}{2+s}$

From the first property, we know that the Laplace transform of the impulse response must have the form

$$H(s) = C + \frac{D}{s+2} = C\,\frac{s+2+\frac{D}{C}}{s+2}$$

From the second property, we know that the zero must be at $s = 2$. Therefore $D = -4C$, and

$$H(s) = C\,\frac{s-2}{s+2}\,.$$

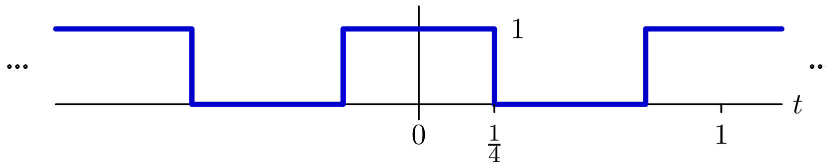
From the third property, we know that the DC gain is $+1$. Therefore $C = -1$, and

$$H(s) = \frac{2-s}{2+s}$$

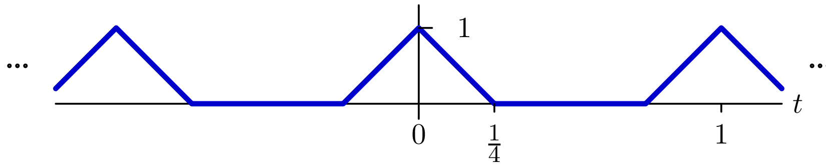
3. Input/Output Pairs *[18 points]*

The following signals are all periodic with period $T = 1$.

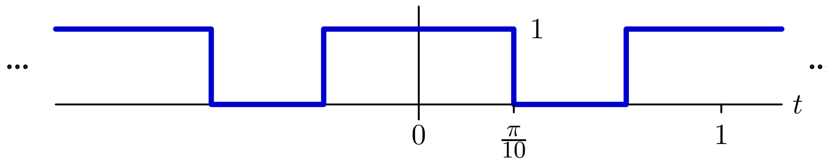
$x_1(t) = x_1(t+1)$



$x_2(t) = x_2(t+1)$



$x_3(t) = x_3(t+1)$



Indicate which of the systems on the next page could/could not be linear and time-invariant.

Grading: +3 for each correct answer; −3 for each incorrect answer; 0 for blank or **?**.

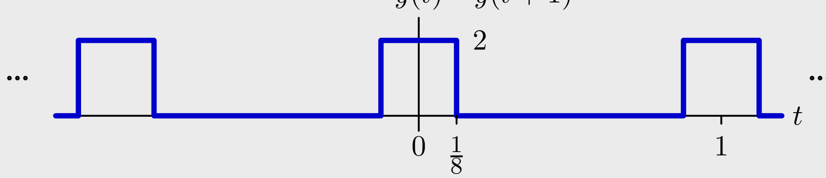
We can use the "filter" idea as follows. First calculate the Fourier series coefficients. Then ask if each Fourier series coefficient in the output is a scaled version of the corresponding coefficient in the input.

$$x_1(t) \leftrightarrow a_k = \frac{1}{1}\int_{\frac{-1}{4}}^{\frac{1}{4}} e^{-j\frac{2\pi}{1}kt}dt = \frac{\sin\frac{\pi k}{2}}{\pi k} = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{\pi k} & |k| = 1, 5, 9, 13, \ldots \\ -\frac{1}{\pi k} & |k| = 3, 7, 11, 15, \ldots \\ 0 & |k| = 2, 4, 6, 8, \ldots \end{cases}$$

$$x_2(t) = y(t) \underset{T}{*} y(t)$$

where $y(t)$ is the following signal:

$y(t) = y(t+1)$



$$y(t) \leftrightarrow d_k = \frac{1}{1}\int_{\frac{-1}{8}}^{\frac{1}{8}} 2e^{-j\frac{2\pi}{1}kt}dt = \frac{2\sin\frac{\pi k}{4}}{\pi k}$$

$$x_2(t) \leftrightarrow b_k = Td_k^2 = 1 \times \frac{4\sin^2\frac{\pi k}{4}}{\pi^2 k^2} = \begin{cases} \frac{1}{4} & k = 0 \\ \frac{2}{\pi^2 k^2} & |k| = 1, 3, 5, 7, 9, 11, 13, \ldots \\ \frac{4}{\pi^2 k^2} & |k| = 2, 6, 10, 14, \ldots \\ 0 & |k| = 4, 8, 12, 16, \ldots \end{cases}$$

$$x_3(t) \leftrightarrow c_k = \frac{1}{1}\int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} e^{-j\frac{2\pi}{1}kt}dt = \frac{\sin\frac{2\pi^2 k}{10}}{\pi k}$$

$x_1(t) \rightarrow$ System #1 $\rightarrow x_2(t)$ System #1 could be LTI? (**yes**/**no**): no

The Fourier series coefficients at $k = 2, 6, 10, \ldots$ are zero in x_1 but these are not zero in x_2. Therefore the system could not be LTI.

$x_1(t) \rightarrow$ System #2 $\rightarrow x_3(t)$ System #2 could be LTI? (**yes**/**no**): no

The Fourier series coefficients at $k = 2, 4, 6, 8, 10, \ldots$ are zero in x_1 but these are not zero in x_3. Therefore the system could not be LTI.

$x_2(t) \rightarrow$ System #3 $\rightarrow x_1(t)$ System #3 could be LTI? (**yes**/**no**): yes

All of the nonzero Fourier coefficients in x_1 are also present in x_2. Therefore the system could be LTI.

$x_2(t) \rightarrow$ System #4 $\rightarrow x_3(t)$ System #4 could be LTI? (**yes**/**no**): no

The Fourier series coefficients at $k = 4, 8, 12, 16, \ldots$ are zero in x_2 but these are not zero in x_3. Therefore the system could not be LTI.

$x_3(t) \rightarrow$ System #5 $\rightarrow x_1(t)$ System #5 could be LTI? (**yes**/**no**): yes

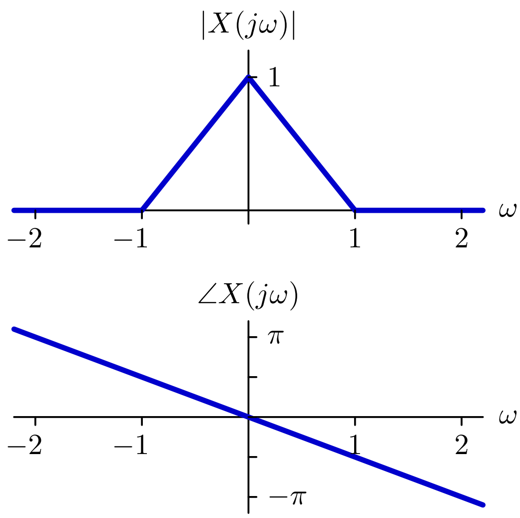
All of the nonzero Fourier coefficients in x_1 are also present in x_3. Therefore the system could be LTI.

$x_3(t) \rightarrow$ System #6 $\rightarrow x_2(t)$ System #6 could be LTI? (**yes**/**no**): yes

All of the nonzero Fourier coefficients in x_2 are also present in x_3. Therefore the system could be LTI.

4. Fourier Transforms *[25 points]*

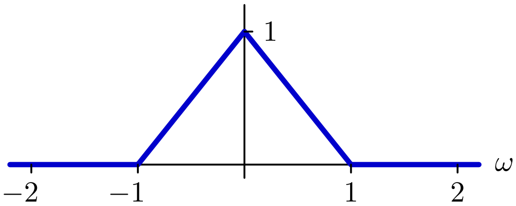
The magnitude and angle of the Fourier transform of a signal $x(t)$ are given in the following plots.



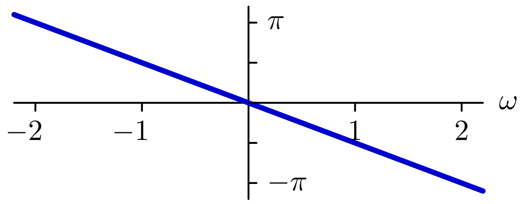
Five signals are derived from $x(t)$ as shown in the left column of the following table. Six magnitude plots (M1-M6) and six angle plots (A1-A6) are shown on the next page. Determine which of these plots is associated with each of the derived signals and place the appropriate label (e.g., M1 or A3) in the following table. Note that more than one derived signal could have the same magnitude or angle.

signal	magnitude	angle
$\frac{dx(t)}{dt}$	M5	A4
$(x * x)(t)$	M3	A2
$x\left(t-\frac{\pi}{2}\right)$	M1	A2
$x(2t)$	M4	A3
$x^2(t)$	M6	A1

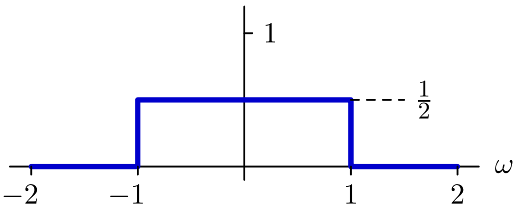
M1



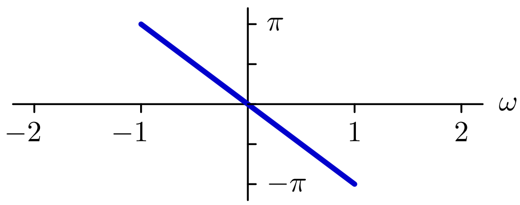
A1



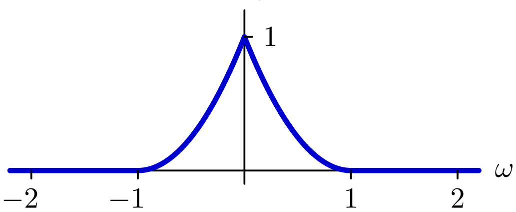
M2



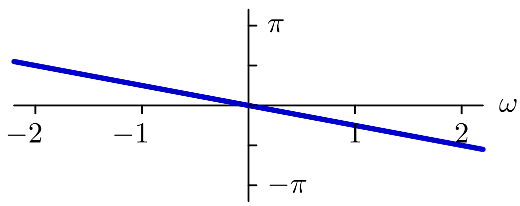
A2



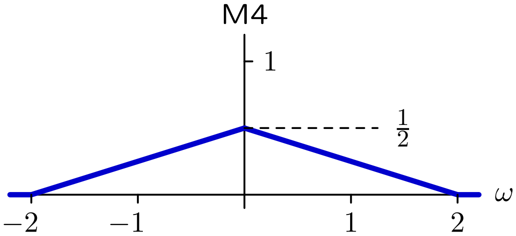
M3



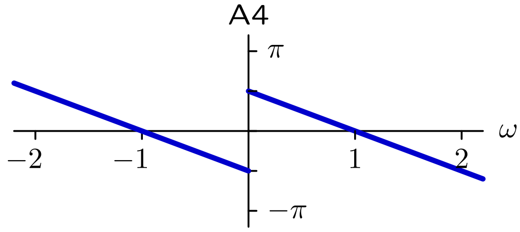
A3



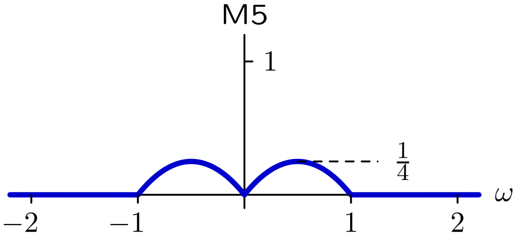
M4



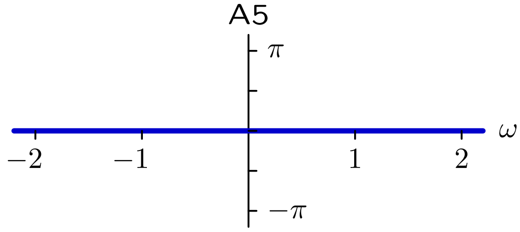
A4



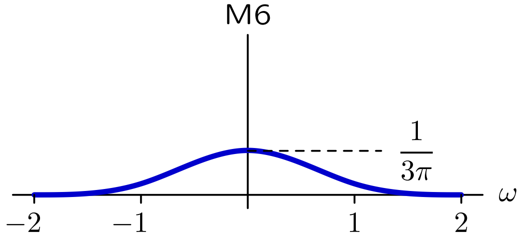
M5



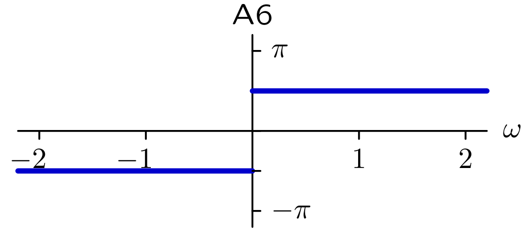
A5



M6



A6

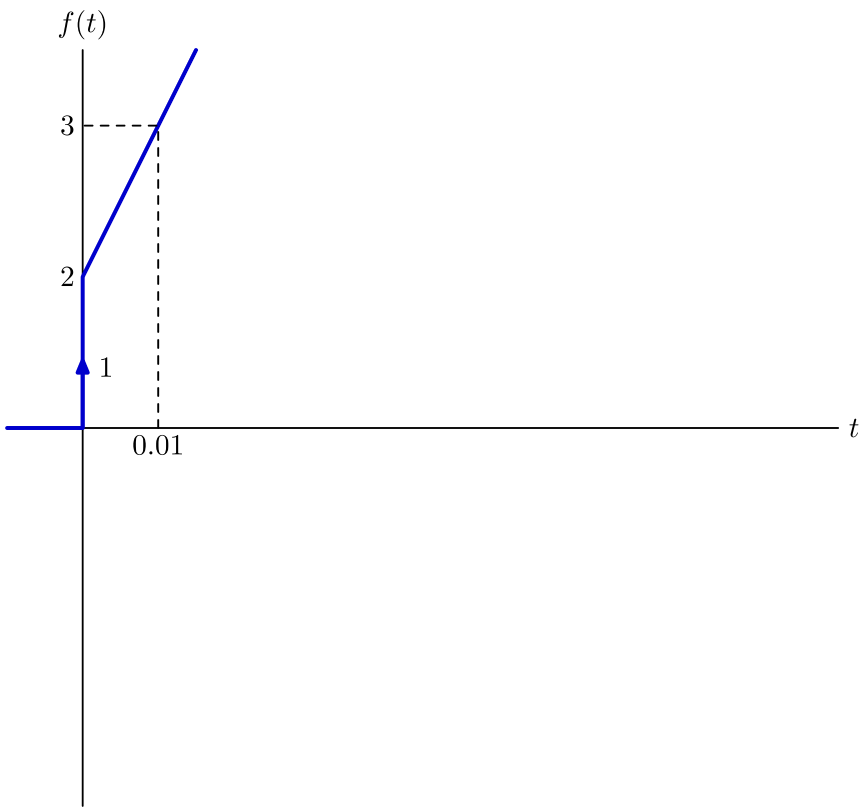


5. Feedback and Control *[25 points]*

Consider a causal LTI system described by $F(s)$ as follows:

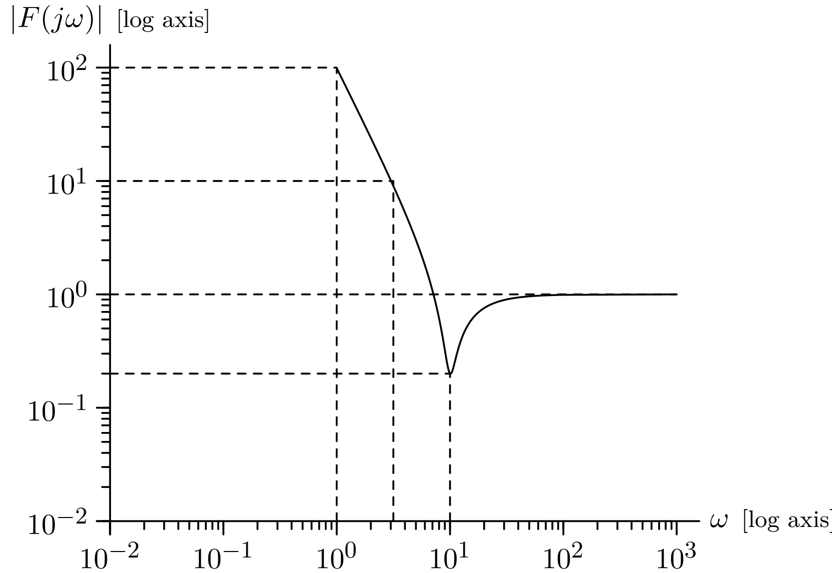
$$F(s) = \frac{s^2 + 2s + 100}{s^2}\,.$$

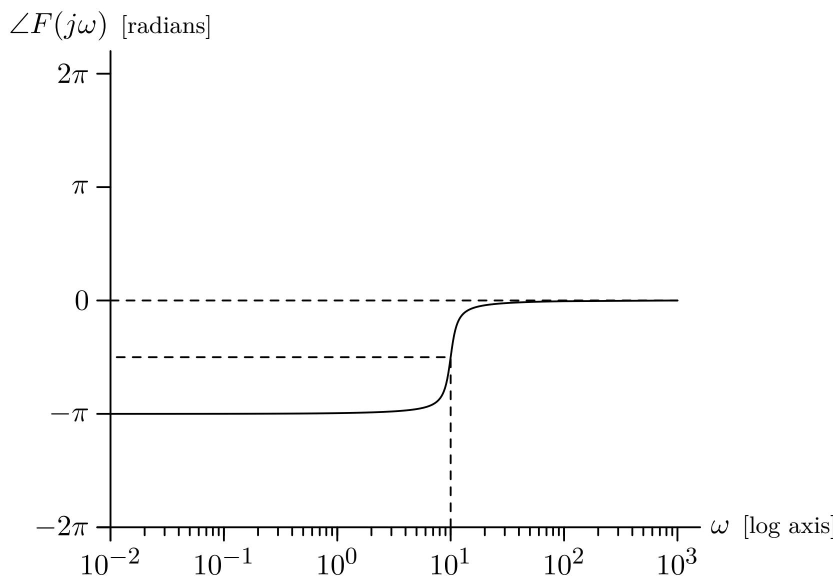
a. Sketch the impulse response $f(t)$ for this system on the axes below. Label the axes and indicate the important features of your plot.



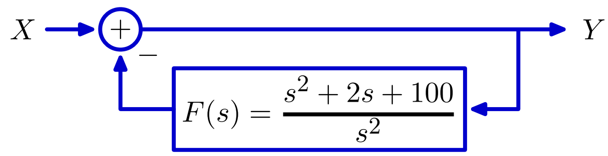
$h(t) = \delta(t) + 2u(t) + 100tu(t)$

b. Sketch the magnitude and angle of $F(j\omega)$ on the following axes. Notice the log axes for ω and for the magnitude. Indicate the important features of your plots, including extreme values.

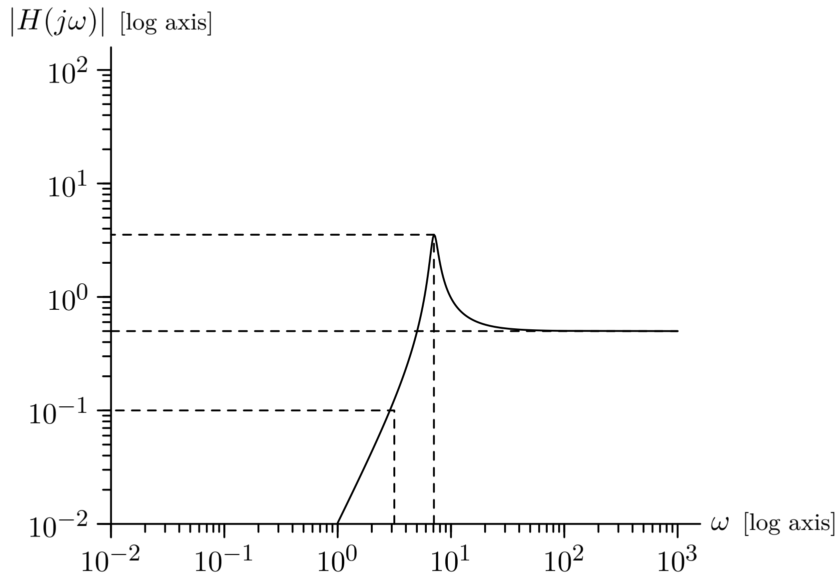


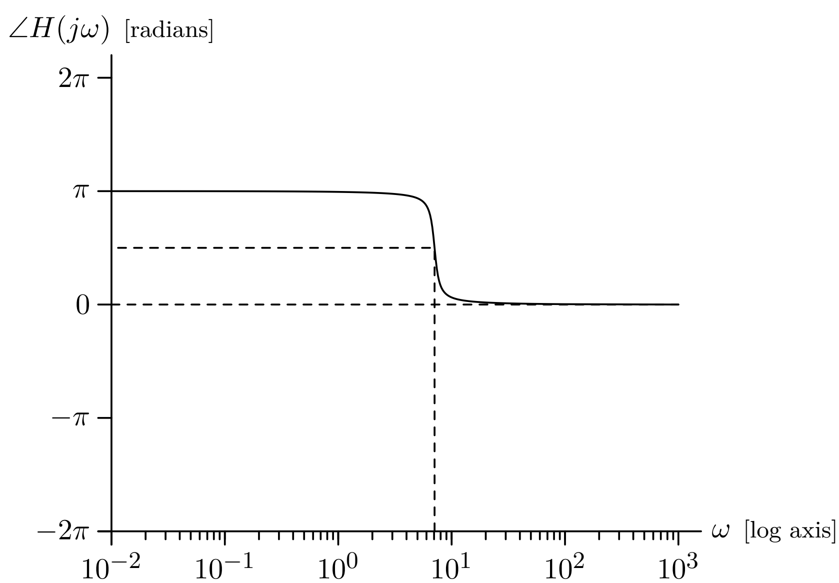


Now consider a feedback system containing $F(s)$ as follows.



c. Let $H(s) = \frac{Y(s)}{X(s)}$ represent the closed-loop system function. Sketch the magnitude and angle of $H(s)$ on the following axes. Notice the log axes for ω and for the magnitude. Indicate the important features of your plots, including extreme values.





$$H(s) = \frac{1}{1 + \frac{s^2+2s+100}{s^2}} = \frac{s^2}{2s^2 + 2s + 100}$$