# Optimization Problems,

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# Relevant Reading for Today's Lecture

- Chapter 13

# The Pros and Cons of Greedy

- Easy to implement
- Computationally efficient

- But does not always yield the best solution
  - Don't even know how good the approximation is

Question 1

# Brute Force Algorithm

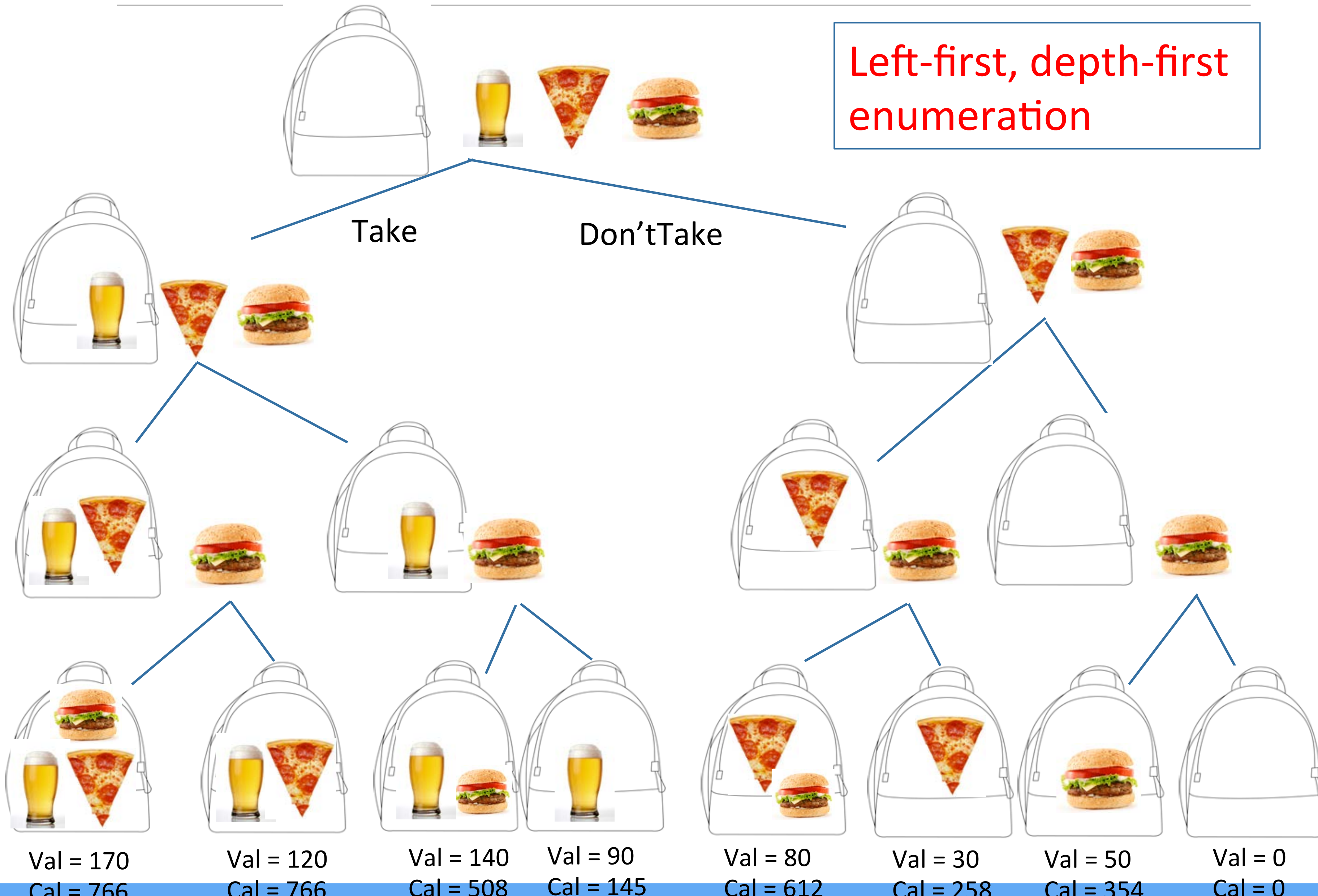
- 1. Enumerate all possible combinations of items.
- 2. Remove all of the combinations whose total units exceeds the allowed weight.
- 3. From the remaining combinations choose any one whose value is the largest.

# Search Tree Implementation

- The tree is built top down starting with the root
- The first element is selected from the still to be considered items
  - If there is room for that item in the knapsack, a node is constructed that reflects the consequence of choosing to take that item. By convention, we draw that as the left child
  - We also explore the consequences of not taking that item. This is the right child
- The process is then applied recursively to non-leaf children
- Finally, chose a node with the highest value that meets constraints

# A Search Tree Enumerates Possibilities

Left-first, depth-first enumeration



# Computational Complexity

- Time based on number of nodes generated
- Number of levels is number of items to choose from
- Number of nodes at level *i* is $2^i$
- So, if there are *n* items the number of nodes is
  - $\sum_{i=0}^{i=n} 2^i$
  - I.e., $O(2^{n+1})$
- An obvious optimization: don't explore parts of tree that violate constraint (e.g., too many calories)
  - Doesn't change complexity
- Does this mean that brute force is never useful?
  - Let's give it a try

# Header for Decision Tree Implementation

```
def maxVal(toConsider, avail):
    """Assumes toConsider a list of items,
               avail a weight
       Returns a tuple of the total value of a
           solution to 0/1 knapsack problem and
           the items of that solution"""
```

`toConsider`. Those items that nodes higher up in the tree (corresponding to earlier calls in the recursive call stack) have not yet considered

`avail`. The amount of space still available

# Body of maxVal (without comments)

```
if toConsider == [] or avail == 0:
        result = (0, ())
    elif toConsider[0].getUnits() > avail:
        result = maxVal(toConsider[1:], avail)
    else:
        nextItem = toConsider[0]
        withVal, withToTake = maxVal(toConsider[1:],
                                     avail - nextItem.getUnits())
        withVal += nextItem.getValue()
        withoutVal, withoutToTake = maxVal(toConsider[1:], avail)
    if withVal > withoutVal:
            result = (withVal, withToTake + (nextItem,))
        else:
            result = (withoutVal, withoutToTake)
    return result
```

Does not actually build search tree

Local variable `result` records best solution found so far

# Try on Example from Lecture 1

- With calorie budget of 750 calories, chose an optimal set of foods from the menu

| Food | wine | beer | pizza | burger | fries | coke | apple | donut |
|---|---|---|---|---|---|---|---|---|
| Value | 89 | 90 | 30 | 50 | 90 | 79 | 90 | 10 |
| calories | 123 | 154 | 258 | 354 | 365 | 150 | 95 | 195 |

# Search Tree Worked Great

- Gave us a better answer
- Finished quickly
- But $2^8$ is not a large number
  - We should look at what happens when we have a more extensive menu to choose from

# Code to Try Larger Examples

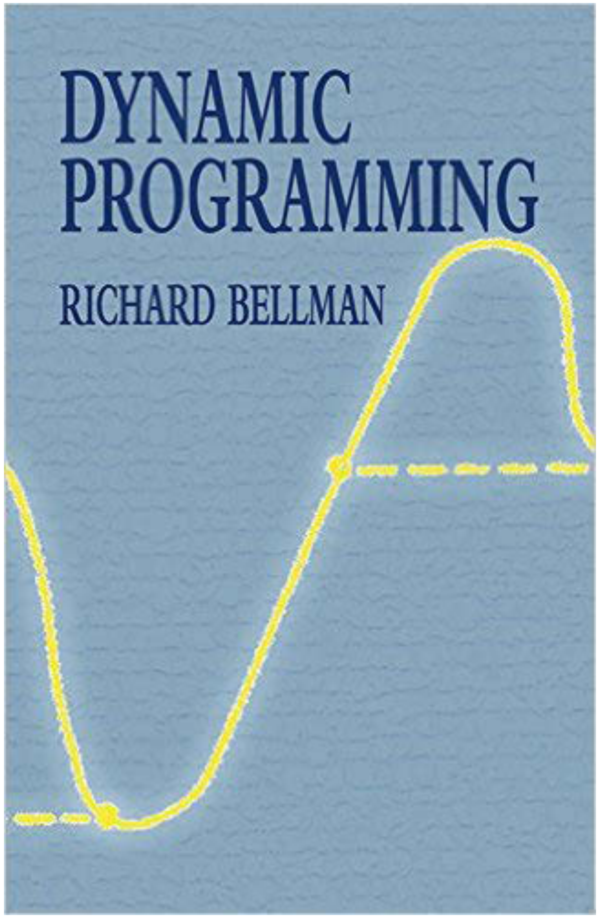
```
import random

def buildLargeMenu(numItems, maxVal, maxCost):
    items = []
    for i in range(numItems):
        items.append(Food(str(i),
                          random.randint(1, maxVal),
                          random.randint(1, maxCost)))
    return items


for numItems in (5,10,15,20,25,30,35,40,45,50,55,60):
    items = buildLargeMenu(numItems, 90, 250)
    testMaxVal(items, 750, False)
```

# Is It Hopeless?

- In theory, yes
- In practice, no!
- Dynamic programming to the rescue



# Dynamic Programming?

Sometimes a name is just a name

“The 1950s were not good years for mathematical research... I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics... What title, what name, could I choose? ... It's impossible to use the word dynamic in a pejorative sense. Try thinking of some combination that will possibly give it a pejorative meaning. It's impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.

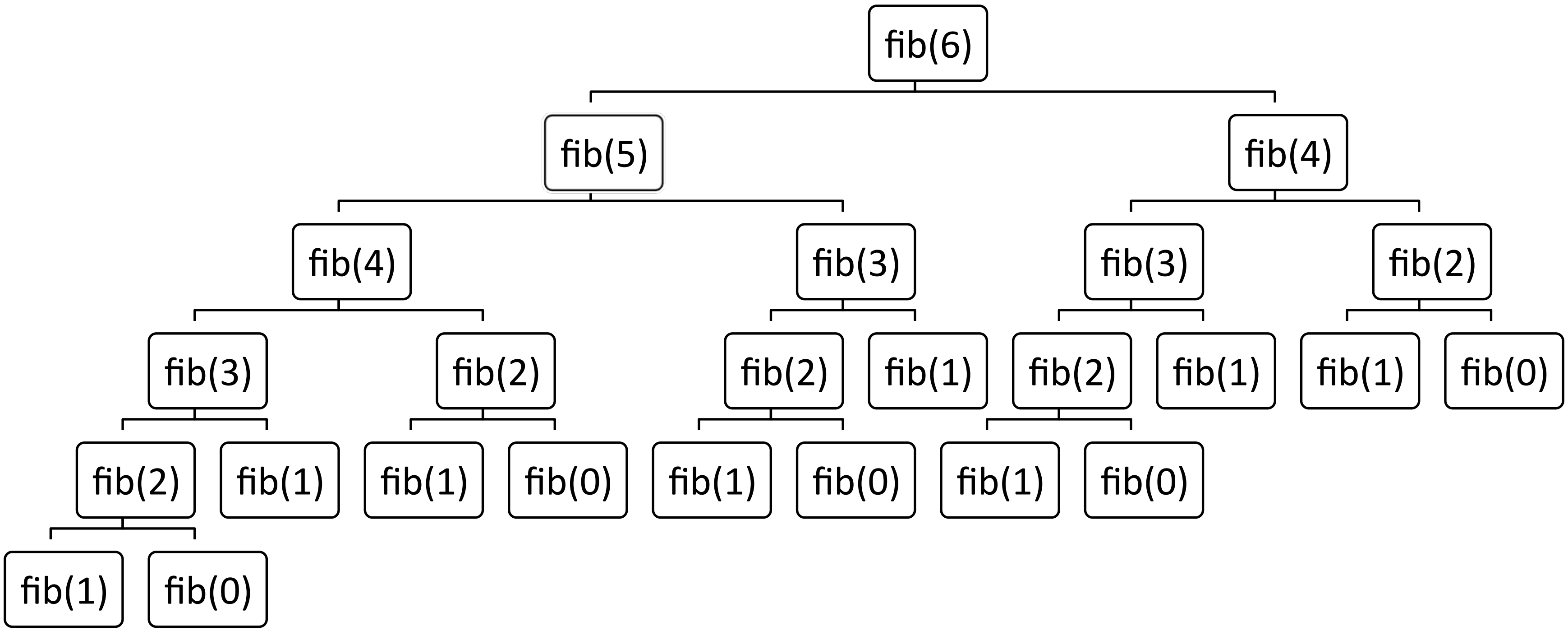
-- Richard Bellman

# Recursive Implementation of Fibonnaci

```
def fib(n):
    if n == 0 or n == 1:
        return 1
    else:
        return fib(n - 1) + fib(n - 2)
```

fib(120) = 8,670,007,398,507,948,658,051,921

# Call Tree for Recursive Fibonnaci(6) = 13



# Clearly a Bad Idea to Repeat Work

- Trade a time for space
- Create a table to record what we've done
  - Before computing fib(x), check if value of fib(x) already stored in the table
    - If so, look it up
    - If not, compute it and then add it to table
  - Called memoization

# Using a Memo to Compute Fibonnaci

```
def fastFib(n, memo = {}):
    """Assumes n is an int >= 0, memo used only by
         recursive calls
       Returns Fibonacci of n"""
    if n == 0 or n == 1:
        return 1
    try:
        return memo[n]
    except KeyError:
        result = fastFib(n-1, memo) +\
                 fastFib(n-2, memo)
        memo[n] = result
        return result
```

# When Does It Work?

- Optimal substructure: a globally optimal solution can be found by combining optimal solutions to local subproblems
  - For x > 1, fib(x) = fib(x - 1) + fib(x – 2)

- Overlapping subproblems: finding an optimal solution involves solving the same problem multiple times
  - Compute fib(x) or many times

# What About 0/1 Knapsack Problem?

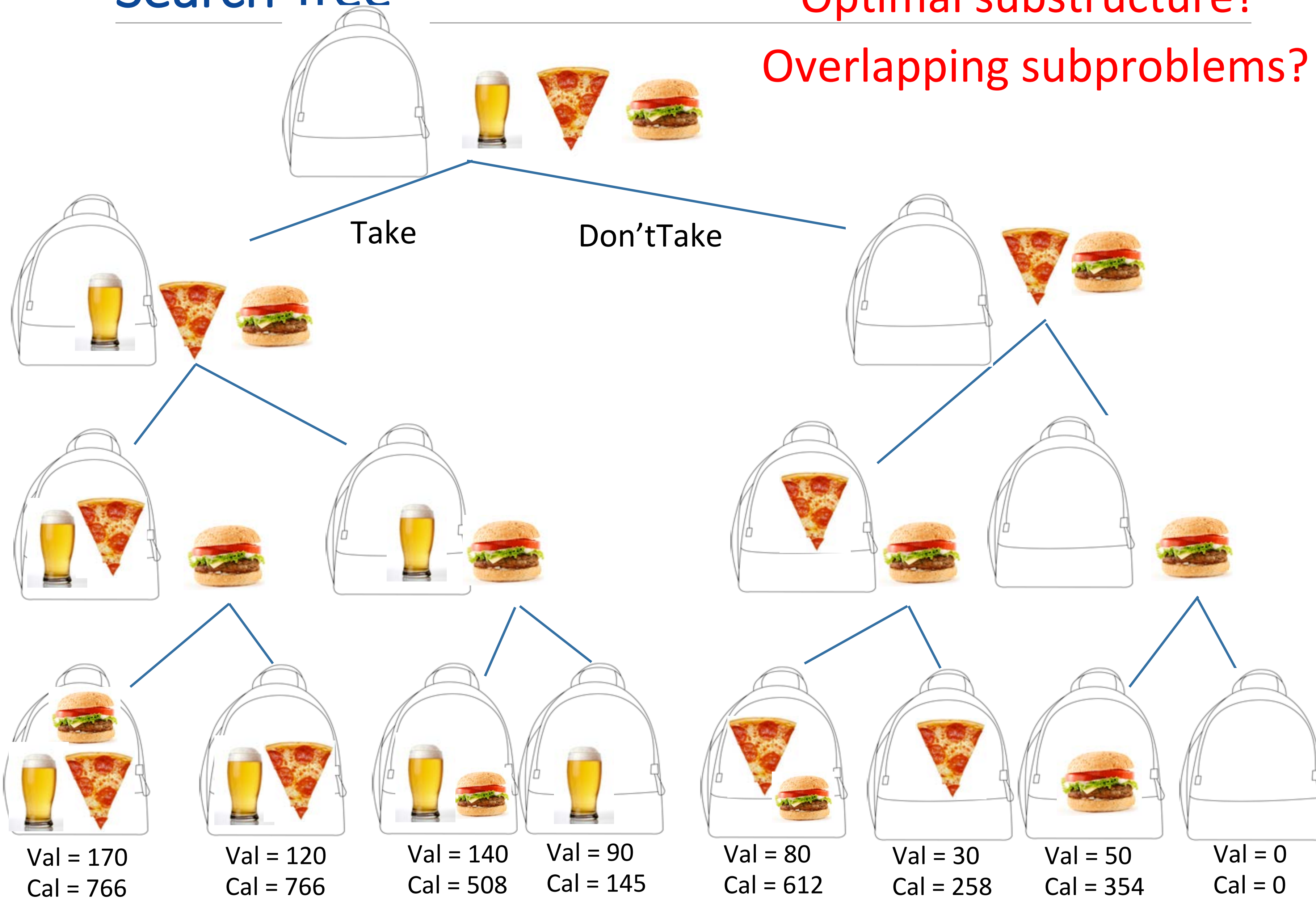
- Do these conditions hold?

Questions 2 and 3

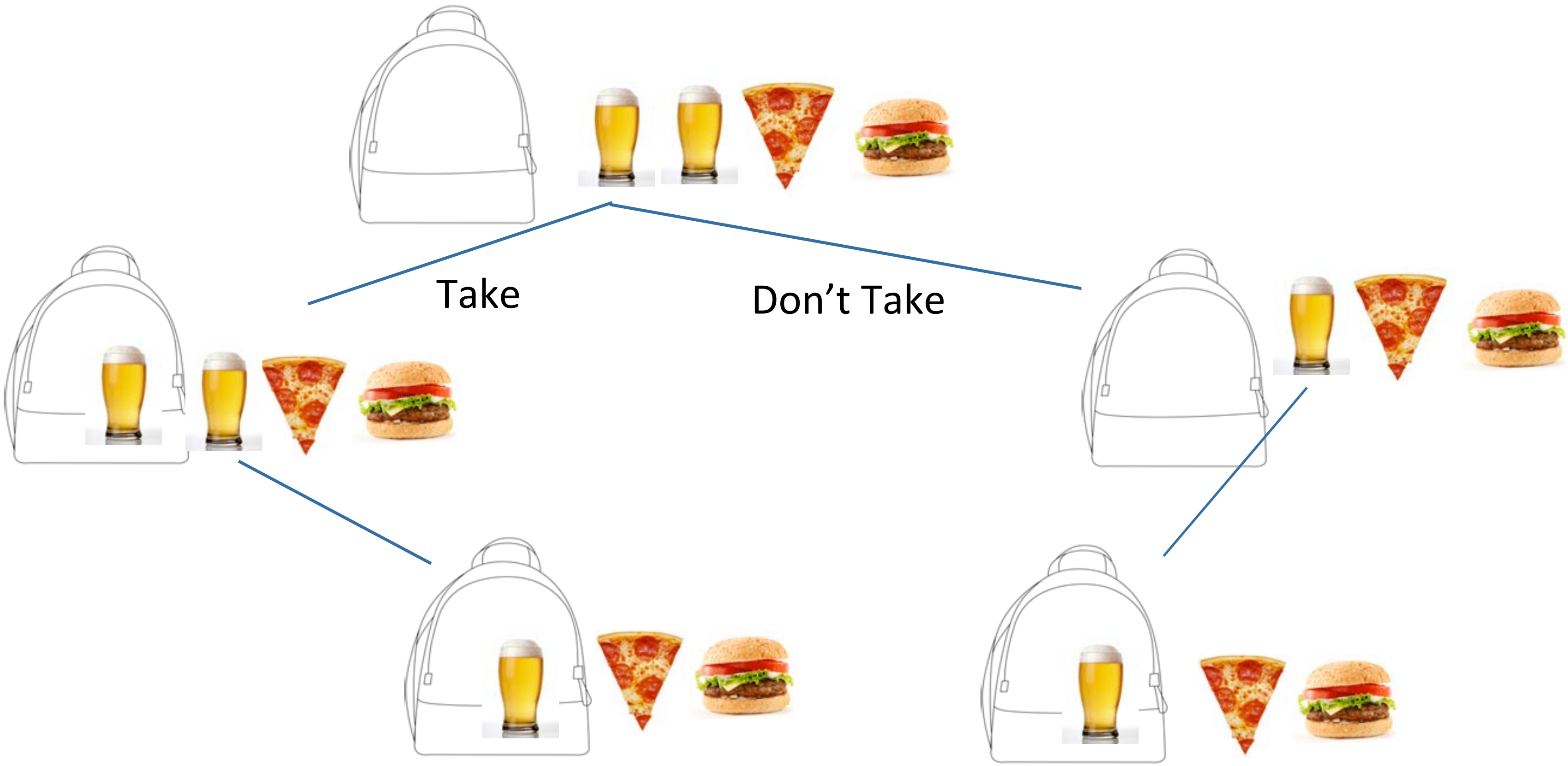
# Search Tree

Optimal substructure?

Overlapping subproblems?



# A Different Menu

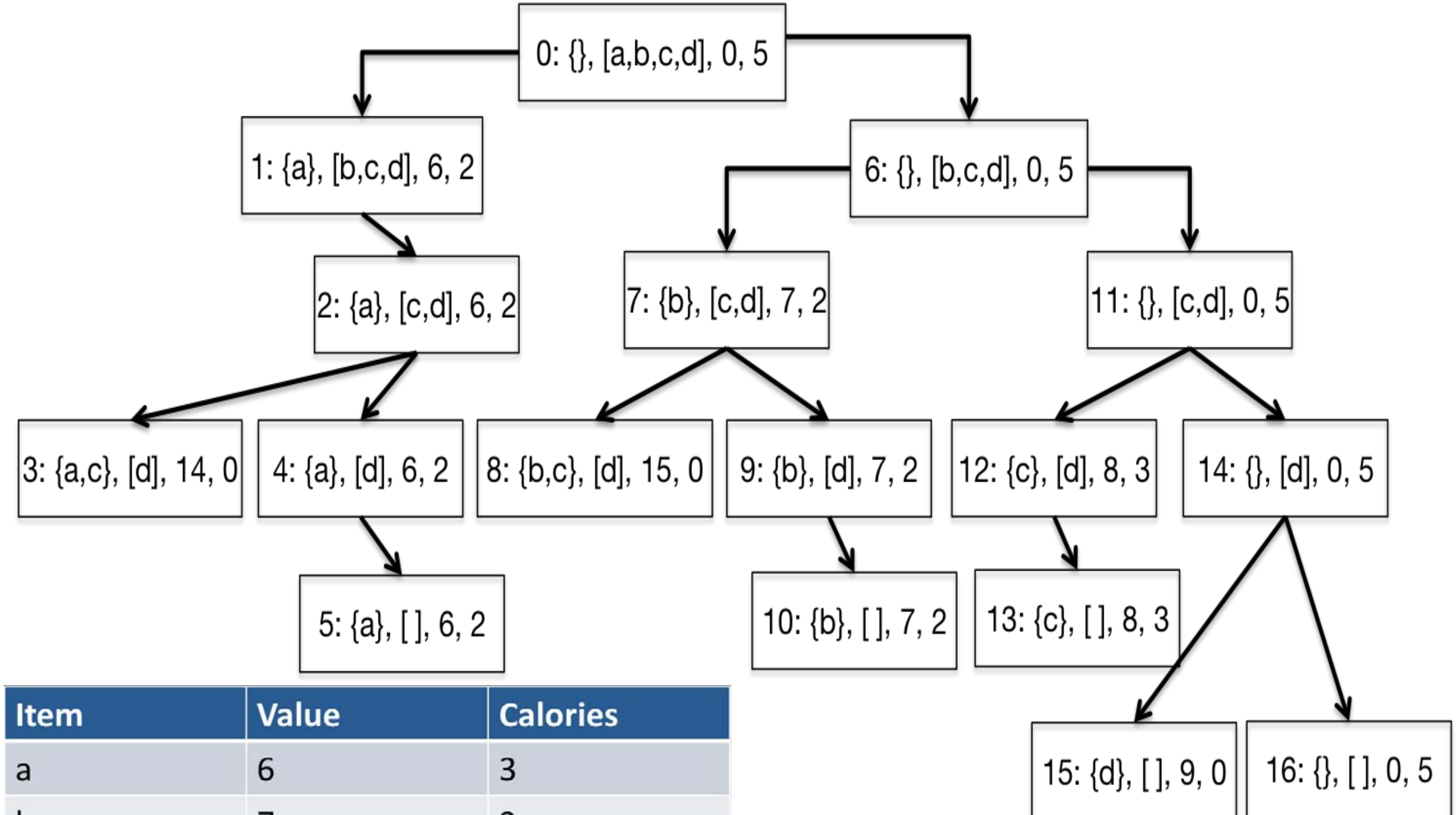


# Need Not Have Copies of Items

| Item | Value | Calories |
| --- | --- | --- |
| a | 6 | 3 |
| b | 7 | 3 |
| c | 8 | 2 |
| d | 9 | 5 |

# Search Tree

- Each node = <taken, left, value, remaining calories>

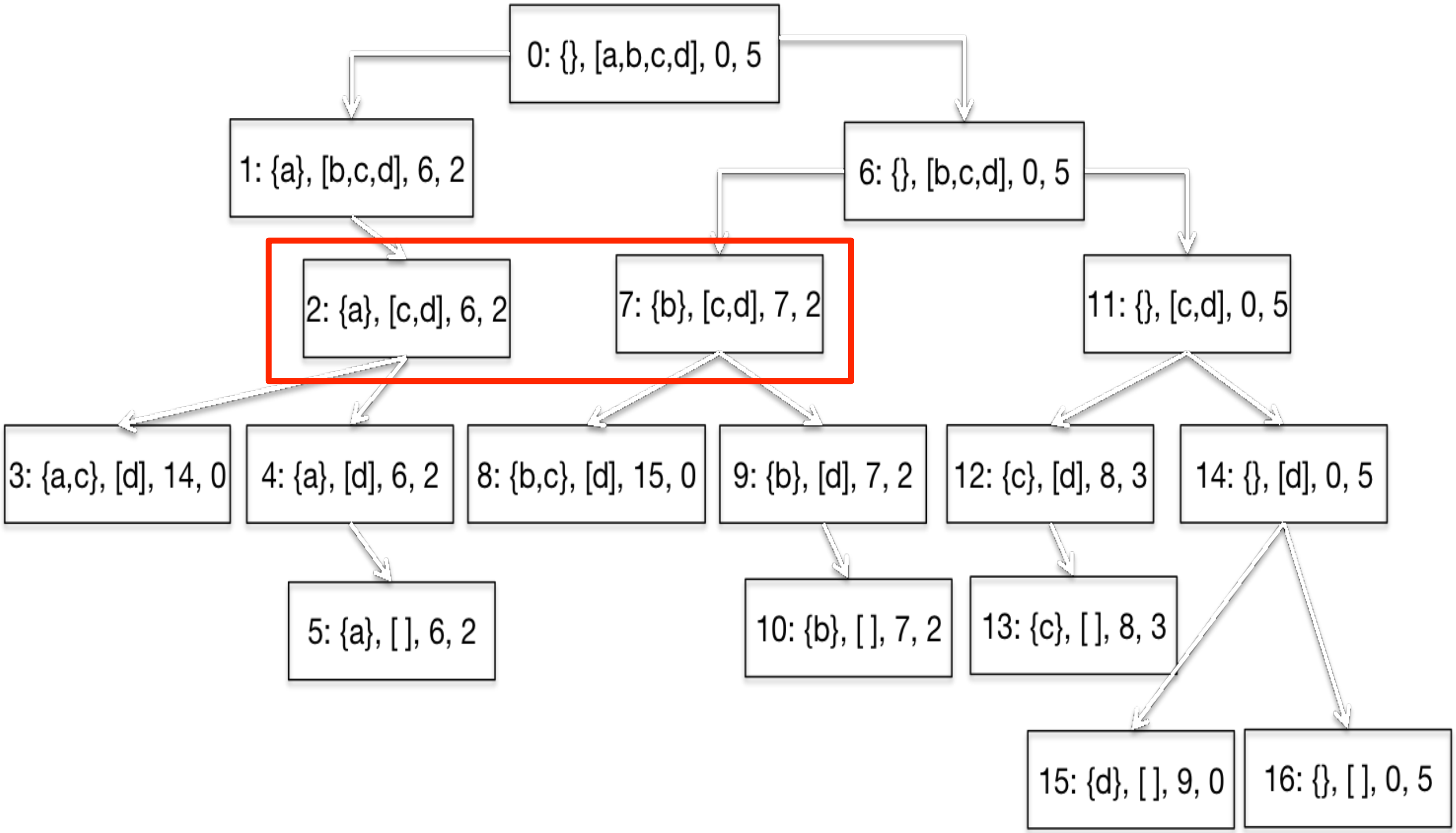



| Item | Value | Calories |
|---|---|---|
| a | 6 | 3 |
| b | 7 | 3 |
| c | 8 | 2 |
| d | 9 | 5 |

# What Problem is Solved at Each Node?

- Given remaining weight, maximize value by choosing among remaining items
- Set of previously chosen items, or even value of that set, doesn't matter!

# Overlapping Subproblems



# Modify maxVal to Use a Memo

- Add memo as a third argument
  - `def fastMaxVal(toConsider, avail, memo = {}):`
- Key of memo is a tuple
  - (items left to be considered, available weight)
  - Items left to be considered represented by `len(toConsider)`
- First thing body of function does is check whether the optimal choice of items given the the available weight is already in the memo
- Last thing body of function does is update the memo

# Performance

| len(items) | 2**len(items) | Number of calls |
|---|---|---|
| 2 | 4 | 7 |
| 4 | 16 | 25 |
| 8 | 256 | 427 |
| 16 | 65,536 | 5,191 |
| 32 | 4,294,967,296 | 22,701 |
| 64 | 18,446,744,073,709,551,616 | 42,569 |
| 128 | Big | 83,319 |
| 256 | Really Big | 176,614 |
| 512 | Ridiculously big | 351,230 |
| 1024 | Absolutely huge | 703,802 |

# How Can This Be?

- Problem is exponential
- Have we overturned the laws of the universe?
- Is dynamic programming a miracle?
- No, but computational complexity can be subtle
- Running time of `fastMaxVal` is governed by number of distinct pairs, `<toConsider, avail>`
  - Number of possible values of `toConsider` bounded by `len(items)`
  - Possible values of `avail` a bit harder to characterize
    - Bounded by number of distinct sums of weights
  - Covered in more detail in assigned reading

# Summary of Lectures 1-2

- Many problems of practical importance can be formulated as optimization problems
- Greedy algorithms often provide adequate (though not necessarily optimal) solutions
- Finding an optimal solution is usually exponentially hard
- But dynamic programming often yields good performance for a subclass of optimization problems—those with optimal substructure and overlapping subproblems
  - Solution always correct
  - Fast under the right circumstances

# The "Roll-over" Optimization Problem

$$\text{Score} = ((60 - (a+b+c+d+e))*F + a*ps1 + b*ps2 + c*ps3 + d*ps4 + e*ps5$$

Objective:

Given values for F, ps1, ps2, ps3, ps4, ps5

Find values for a, b, c, d, e that maximize score

Constraints:

a, b, c, d, e are each 10 or 0

$a + b + c + d + e \geq 20$