RECURSION, DICTIONARIES

(download slides and .py files and follow along!)

6.0001 LECTURE 6

QUIZ PREP

- a paper and an online component
- open book/notes
- not open Internet, not open computer
- start printing out whatever you may want to bring

LAST TIME

- tuples - immutable
- lists - mutable
- aliasing, cloning
- mutability side effects

TODAY

- recursion – divide/decrease and conquer
- dictionaries – another mutable object type

RECURSION

Recursion is the process of repeating items in a self-s imilar way.

WHAT IS RECURSION?

- Algorithmically: a way to design solutions to problems by **divide-and-conquer** or **decrease-and-conquer**
    - reduce a problem to simpler versions of the same problem
- Semantically: a programming technique where a **function calls itself**
    - in programming, goal is to NOT have infinite recursion
        - must have **1 or more base cases** that are easy to solve
        - must solve the same problem on **some other input** with the goal of simplifying the larger problem input

ITERATIVE ALGORITHMS SO FAR

- looping constructs (`while` and `for` loops) lead to **iterative** algorithms
- can capture computation in a set of **state variables** that update on each iteration through loop

MULTIPLICATION – ITERATIVE SOLUTION

- "multiply a * b" is equivalent to "add a to itself b times"
- capture **state** by
  - an **iteration** number (`i`) starts at b

    `i` ← `i-1` and stop when 0
  - a current **value of computation** (`result`)

    `result` ← `result + a`

`a + a + a + a + … + a`

0a 1a 2a 3a 4a

```
def mult_iter(a, b):
    result = 0
    while b > 0:
        result += a
        b -= 1
    return result
```

iteration

current value of computation, a running sum

current value of iteration variable

MULTIPLICATION – RECURSIVE SOLUTION

- **recursive step**
  - think how to reduce problem to a **simpler/ smaller version** of same problem
- **base case**
  - keep reducing problem until reach a simple case that can be **solved directly**
  - when b = 1, a*b = a

$$a*b = \underbrace{a + a + a + a + \ldots + a}_{\text{b times}}$$

$$= a + \underbrace{a + a + a + \ldots + a}_{\text{b-1 times}}$$

$$= a + \boxed{a * (b-1)}$$

recursive reduction

```
def mult(a, b):
    if b == 1:
        return a
    else:
        return a + mult(a, b-1)
```

base case

recursive step

FACTORIAL

`n! = n*(n-1)*(n-2)*(n-3)* … * 1`

- for what `n` do we know the factorial?

n = 1 →

```
if n == 1:
    return 1
```

base case

- how to reduce problem? Rewrite in terms of something simpler to reach base case

n*(n-1)! →

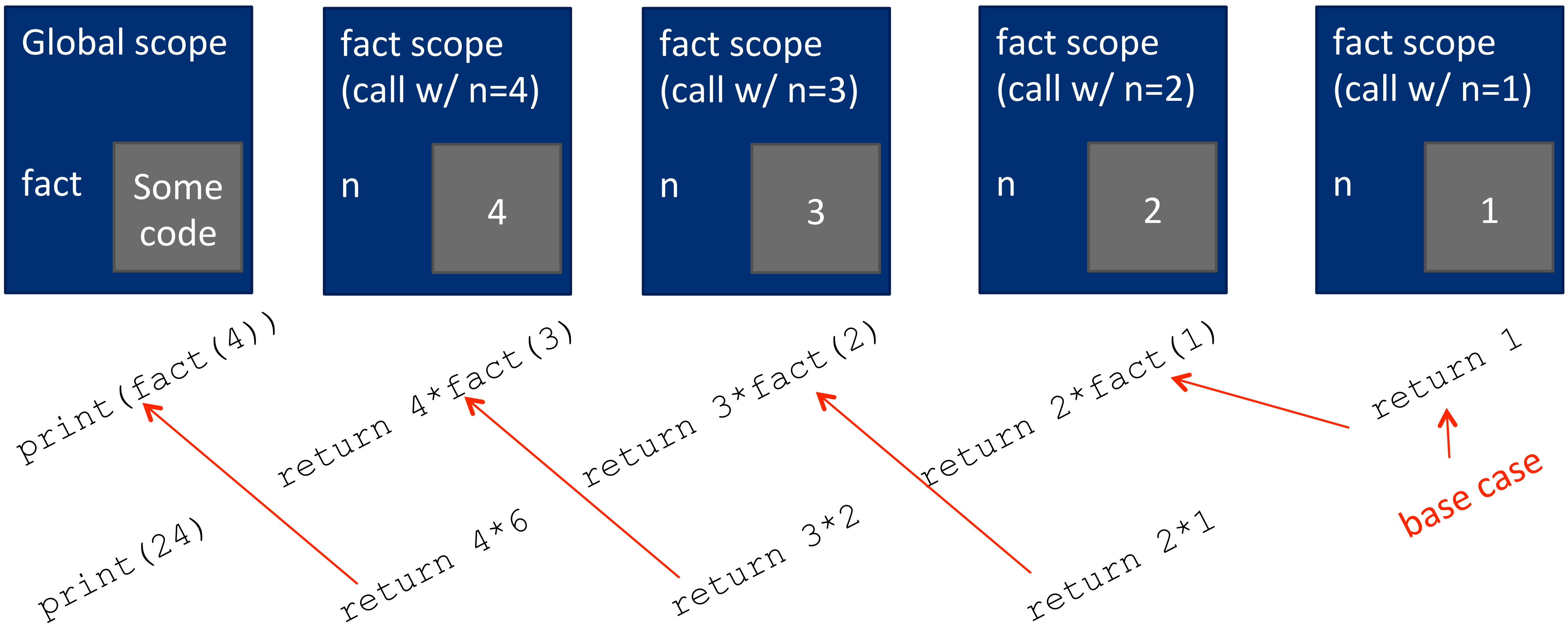
```
else:
    return n*factorial(n-1)
```

recursive step

RECURSIVE FUNCTION SCOPE EXAMPLE

```python
def fact(n):
    if n == 1:
        return 1
    else:
        return n*fact(n-1)

print(fact(4))
```



SOME OBSERVATIONS

- each recursive call to a function creates its **own scope/environment**
- **bindings of variables** in a scope are not changed by recursive call
- flow of control passes back to **previous scope** once function call returns value

using the same variable names but they are different objects in separate scopes

ITERATION vs. RECURSION

```
def factorial_iter(n):
    prod = 1
    for i in range(1,n+1):
        prod *= i
    return prod
```

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n*factorial(n-1)
```

- recursion may be simpler, more intuitive
- recursion may be efficient from programmer POV
- recursion may not be efficient from computer POV

INDUCTIVE REASONING

- How do we know that our recursive code will work?
- `mult_iter` terminates because b is initially positive, and decreases by 1 each time around loop; thus must eventually become less than 1
- `mult` called with b = 1 has no recursive call and stops
- `mult` called with b > 1 makes a recursive call with a smaller version of b; must eventually reach call with b = 1

```
def mult_iter(a, b):
    result = 0
    while b > 0:
        result += a
        b -= 1
    return result


def mult(a, b):
    if b == 1:
        return a
    else:
        return a + mult(a, b-1)
```

MATHEMATICAL INDUCTION

- To prove a statement indexed on integers is true for all values of n:
  - Prove it is true when n is smallest value (e.g. n = 0 or n = 1)
  - Then prove that if it is true for an arbitrary value of n, one can show that it must be true for n+1

EXAMPLE OF INDUCTION

- $0 + 1 + 2 + 3 + \ldots + n = (n(n+1))/2$
- Proof:
  - If n = 0, then LHS is 0 and RHS is 0*1/2 = 0, so true
  - Assume true for some k, then need to show that

    $$0 + 1 + 2 + \ldots + k + (k+1) = ((k+1)(k+2))/2$$

    - LHS is k(k+1)/2 + (k+1) by assumption that property holds for problem of size k
    - This becomes, by algebra, ((k+1)(k+2))/2
  - Hence expression holds for all n >= 0

RELEVANCE TO CODE?

- Same logic applies

```
def mult(a, b):
    if b == 1:
        return a
    else:
        return a + mult(a, b-1)
```

- Base case, we can show that `mult` must return correct answer
- For recursive case, we can assume that `mult` correctly returns an answer for problems of size smaller than b, then by the addition step, it must also return a correct answer for problem of size b
- Thus by induction, code correctly returns answer

TOWERS OF HANOI

- The story:
  - 3 tall spikes
  - Stack of 64 different sized discs – start on one spike
  - Need to move stack to second spike (at which point universe ends)
  - Can only move one disc at a time, and a larger disc can never cover up a small disc

TOWERS OF HANOI

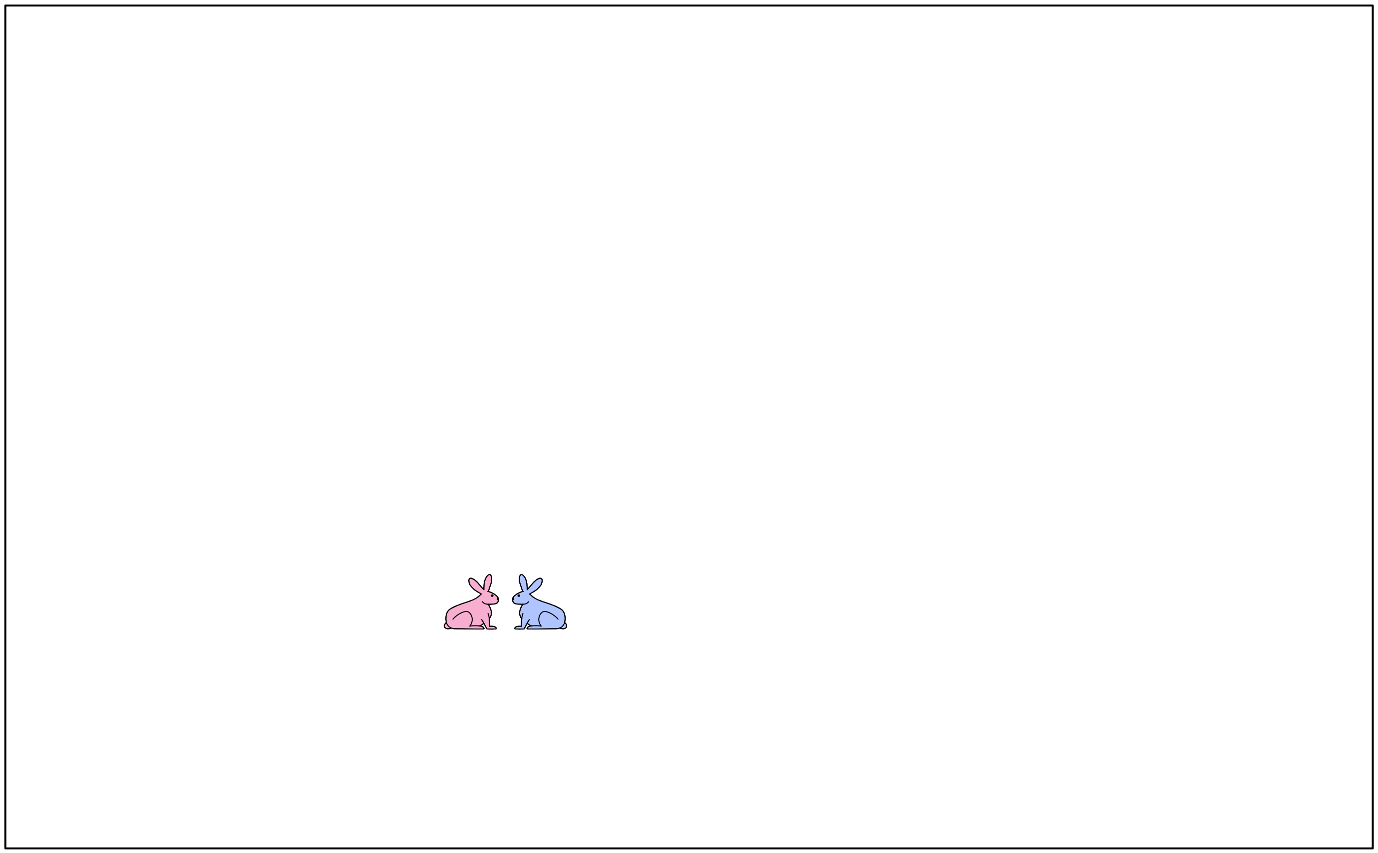
- Having seen a set of examples of different sized stacks, how would you write a program to print out the right set of moves?
- **Think recursively!**
  - Solve a smaller problem
  - Solve a basic problem
  - Solve a smaller problem

```
def printMove(fr, to):
    print('move from ' + str(fr) + ' to ' + str(to))

def Towers(n, fr, to, spare):
    if n == 1:
        printMove(fr, to)
    else:
        Towers(n-1, fr, spare, to)
        Towers(1, fr, to, spare)
        Towers(n-1, spare, to, fr)
```

RECURSION WITH MULTIPLE BASE CASES

- Fibonacci numbers
  - Leonardo of Pisa (aka Fibonacci) modeled the following challenge
    - Newborn pair of rabbits (one female, one male) are put in a pen
    - Rabbits mate at age of one month
    - Rabbits have a one month gestation period
    - Assume rabbits never die, that female always produces one new pair (one male, one female) every month from its second month on.
    - How many female rabbits are there at the end of one year?



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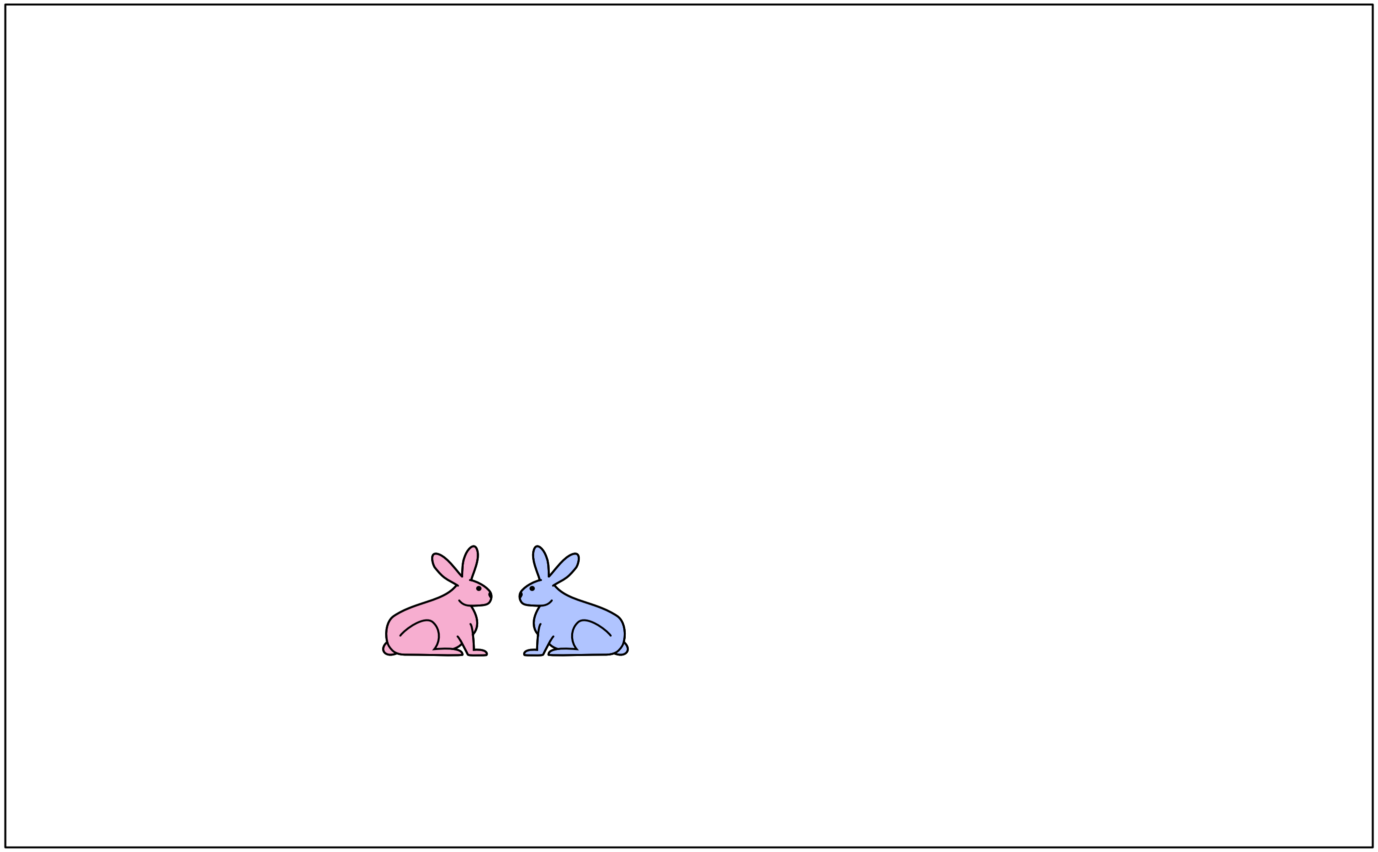

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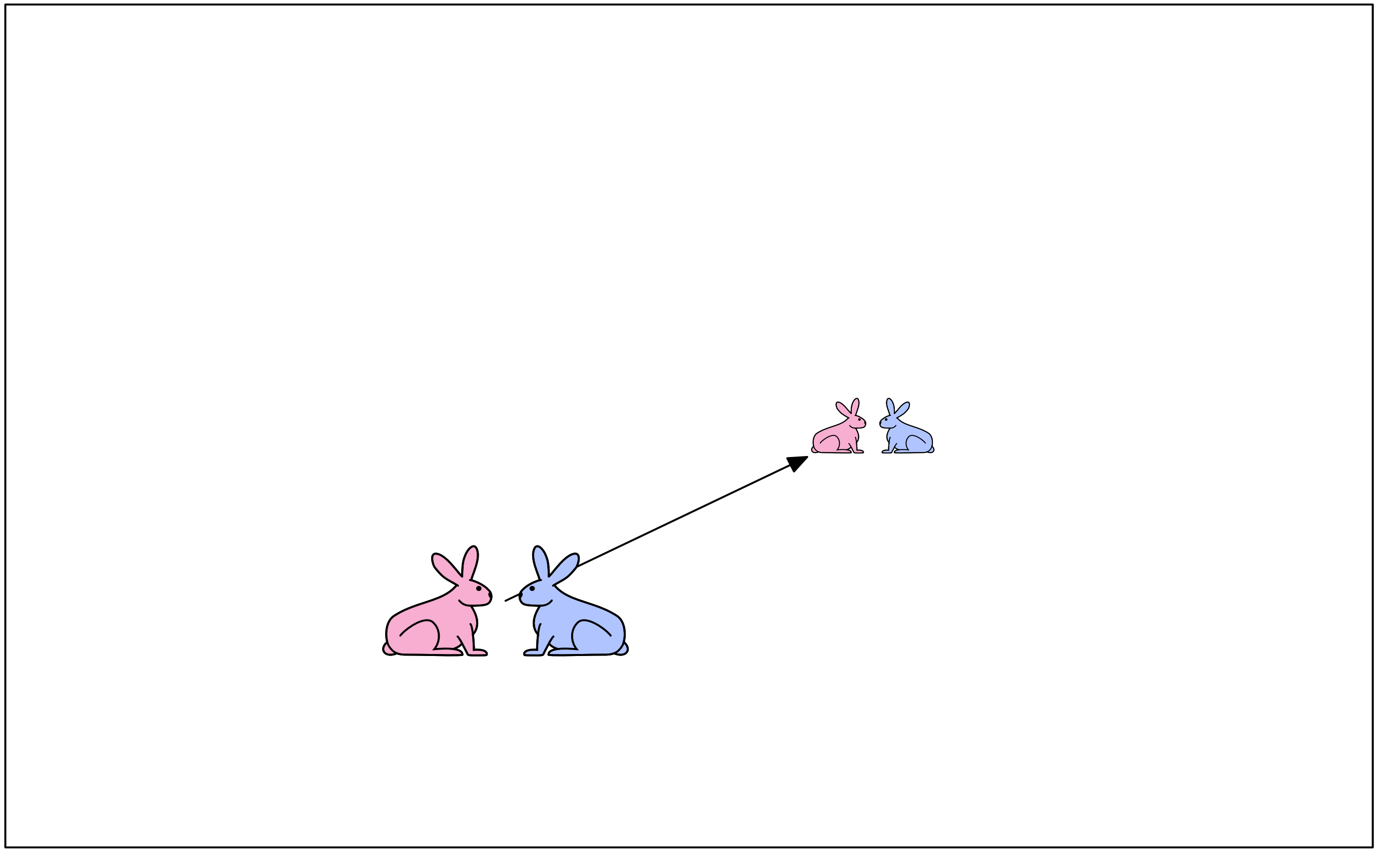

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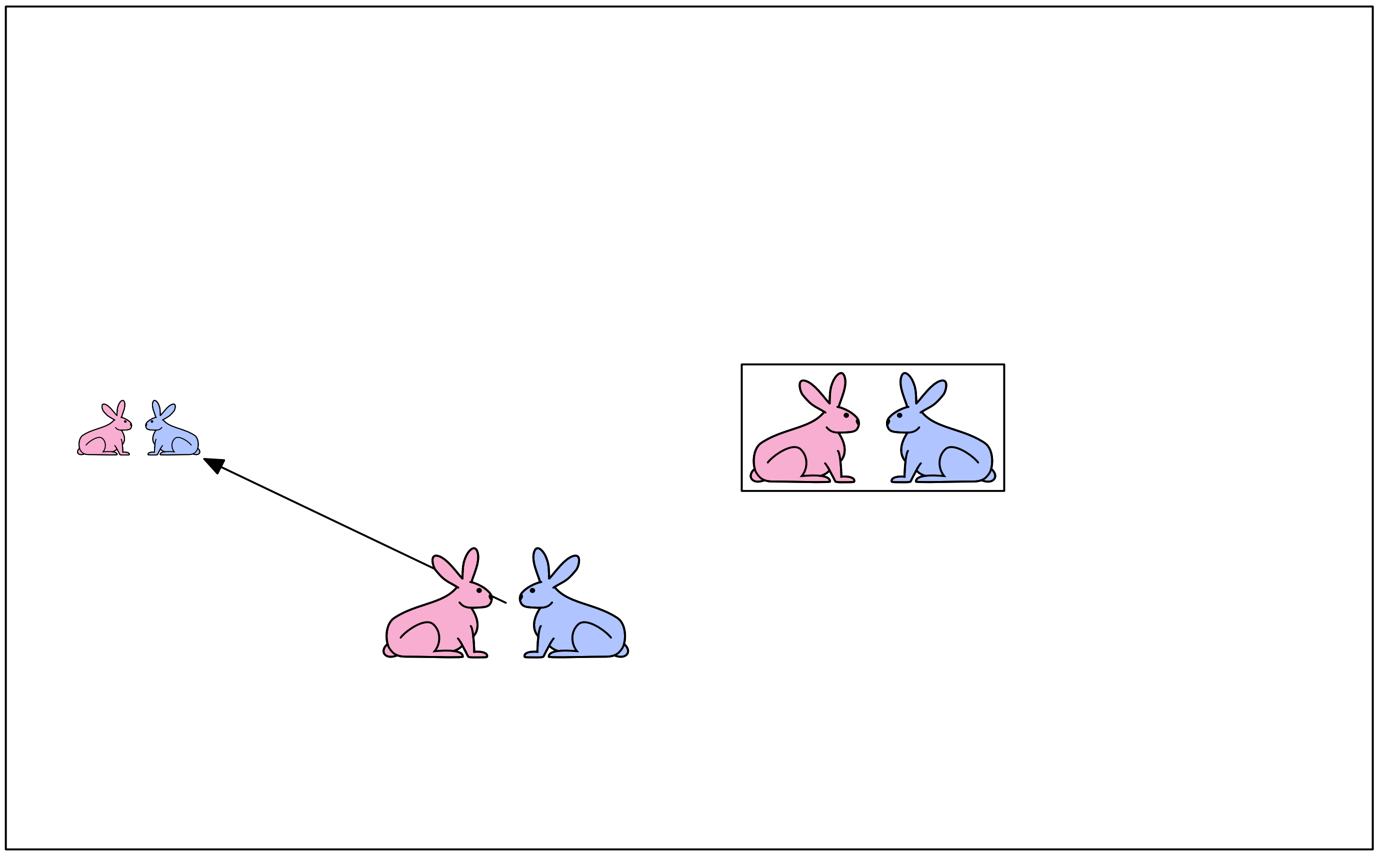

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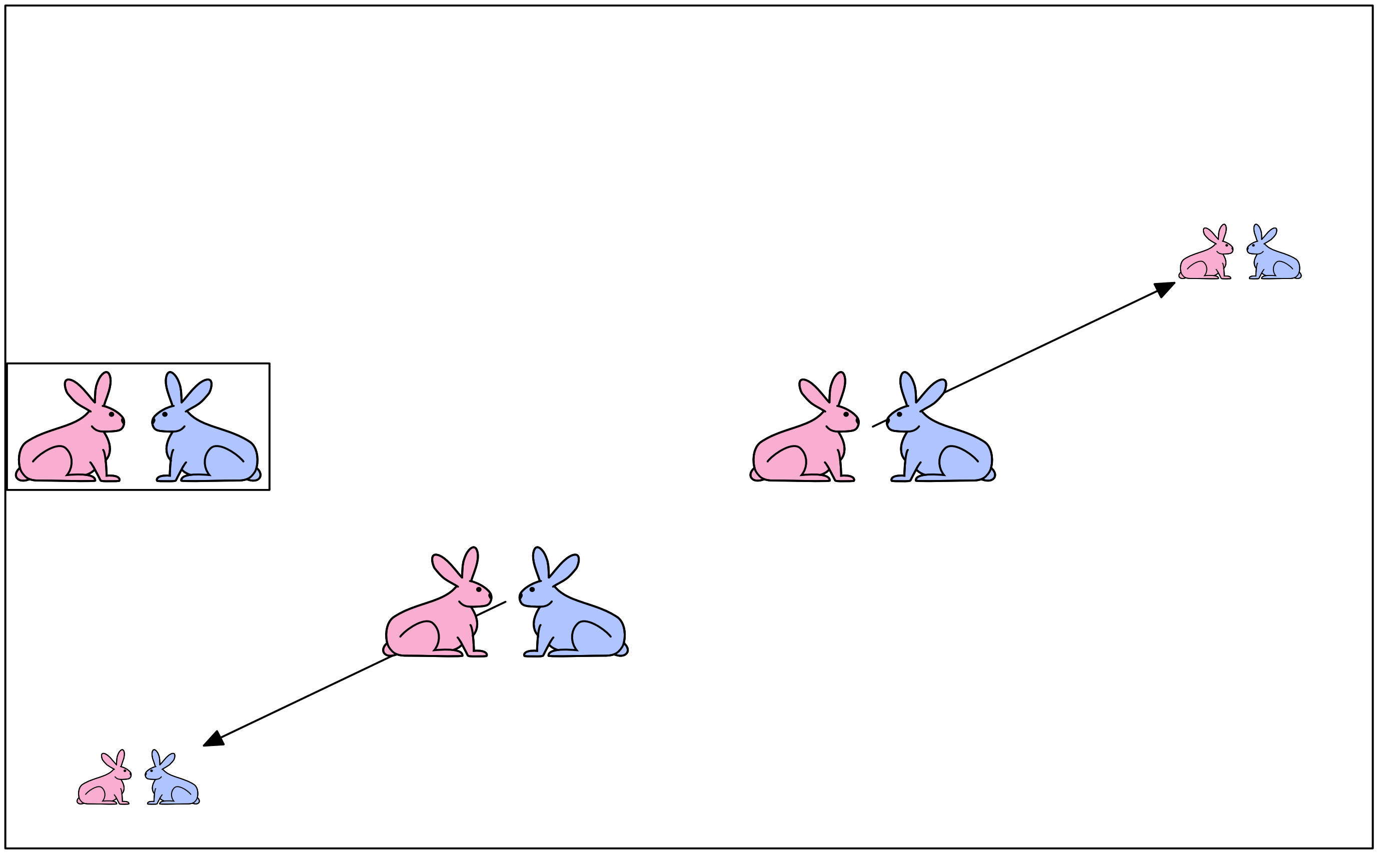

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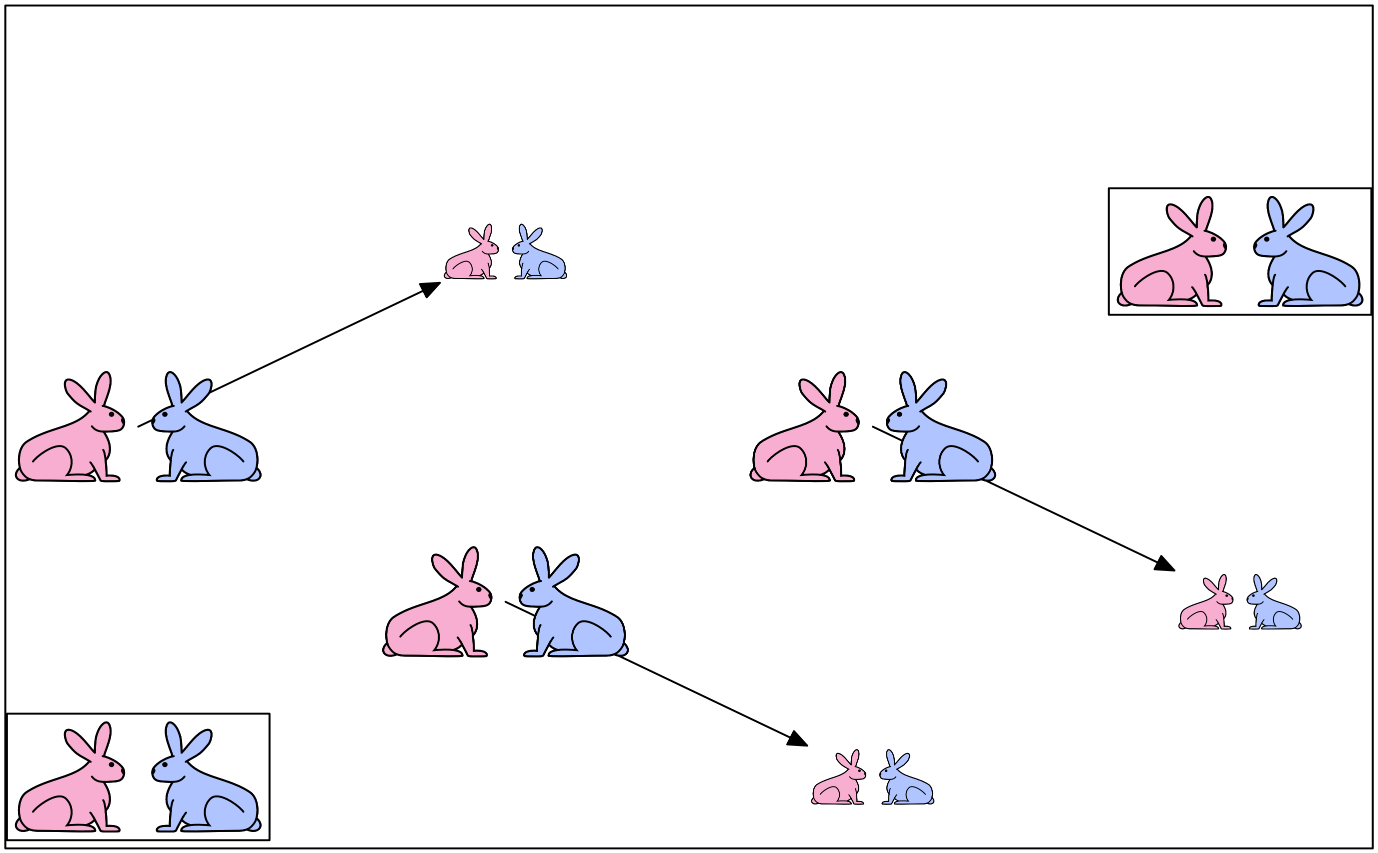

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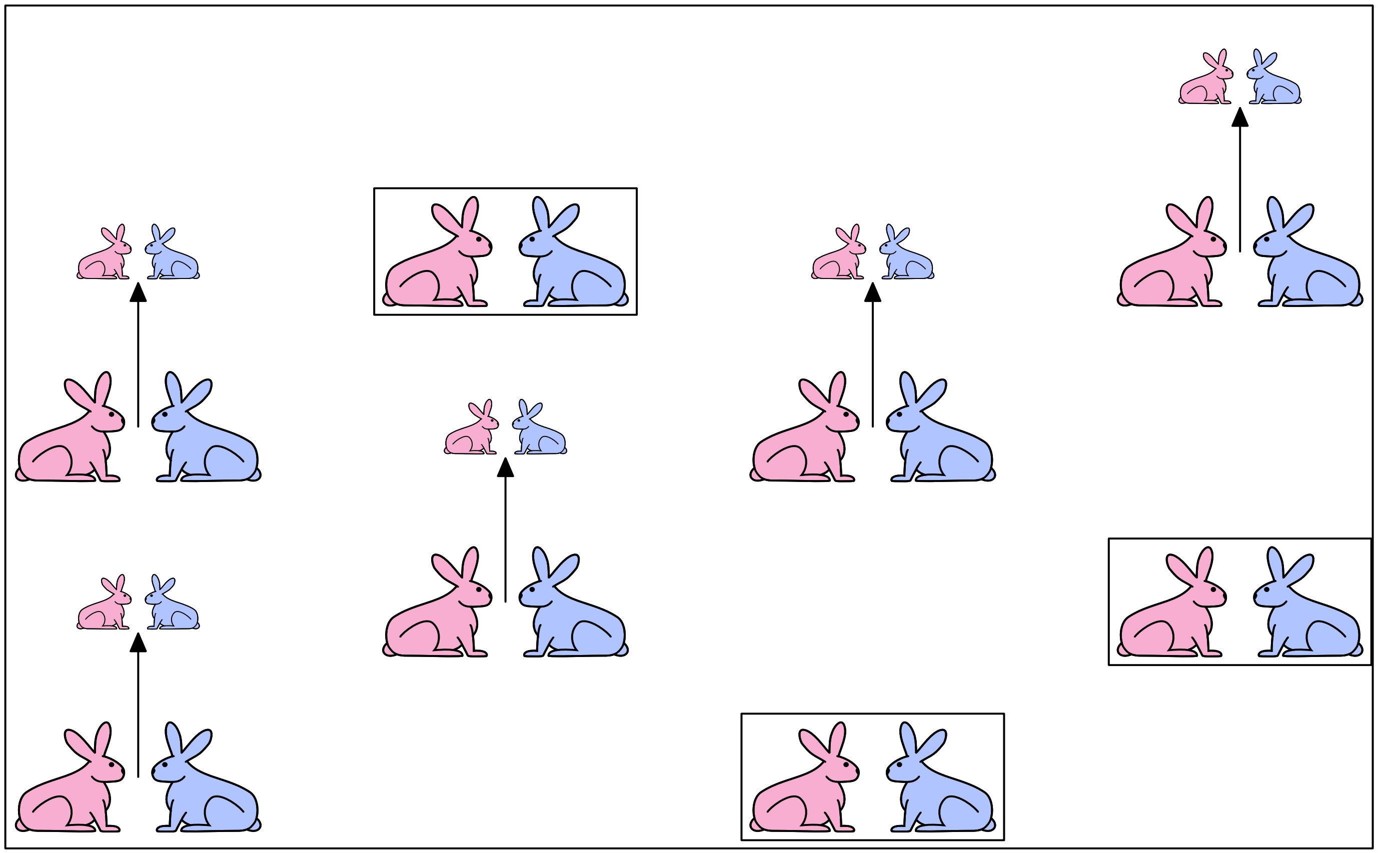

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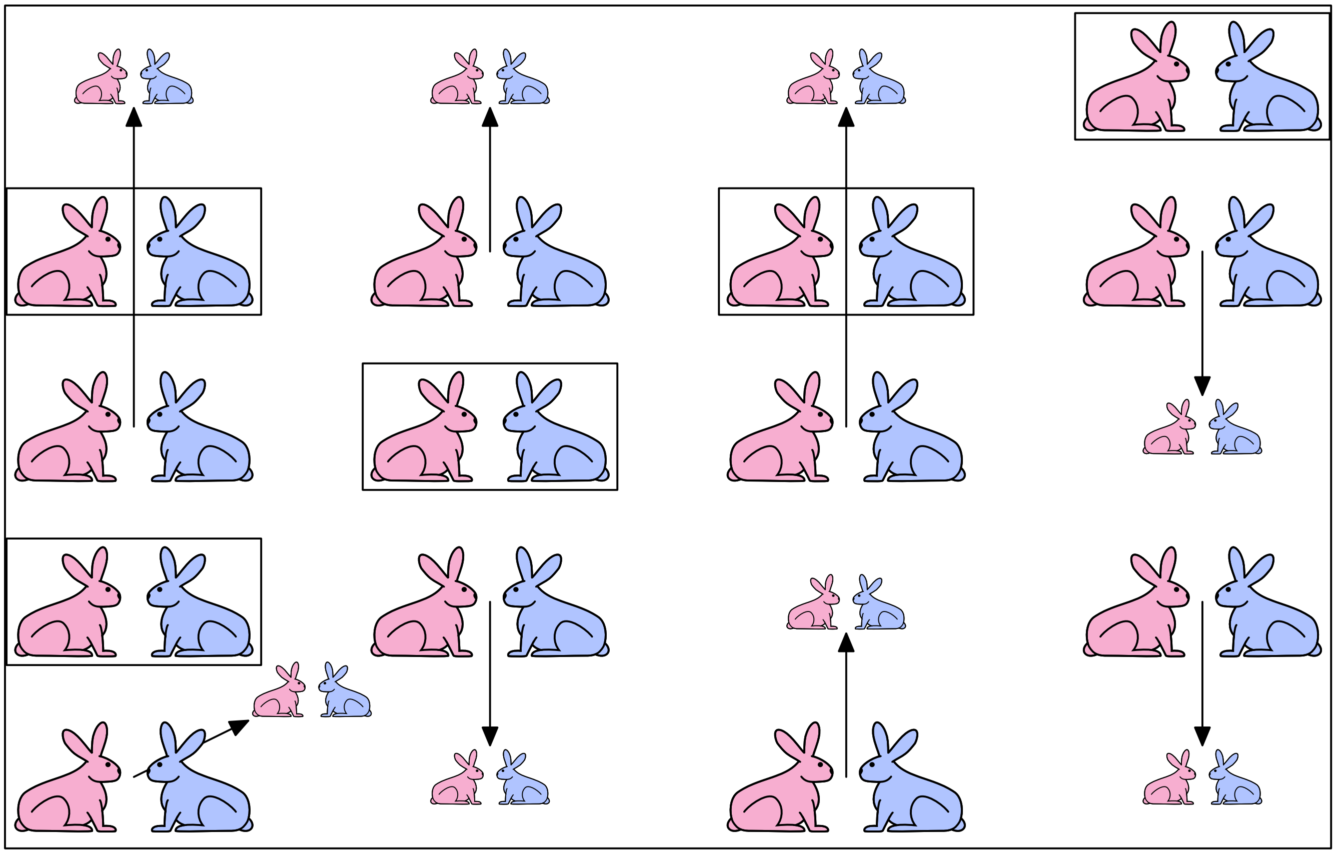

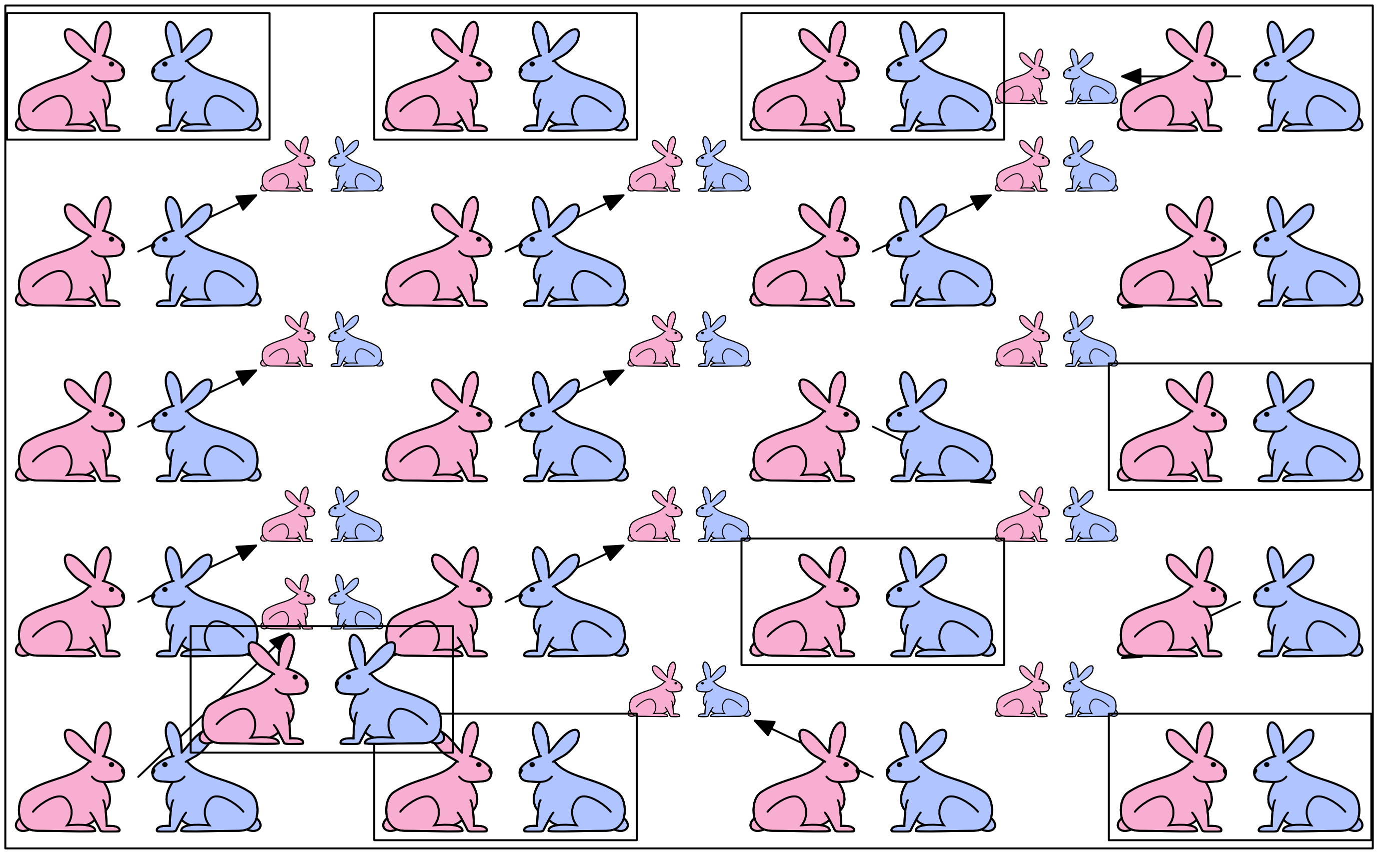

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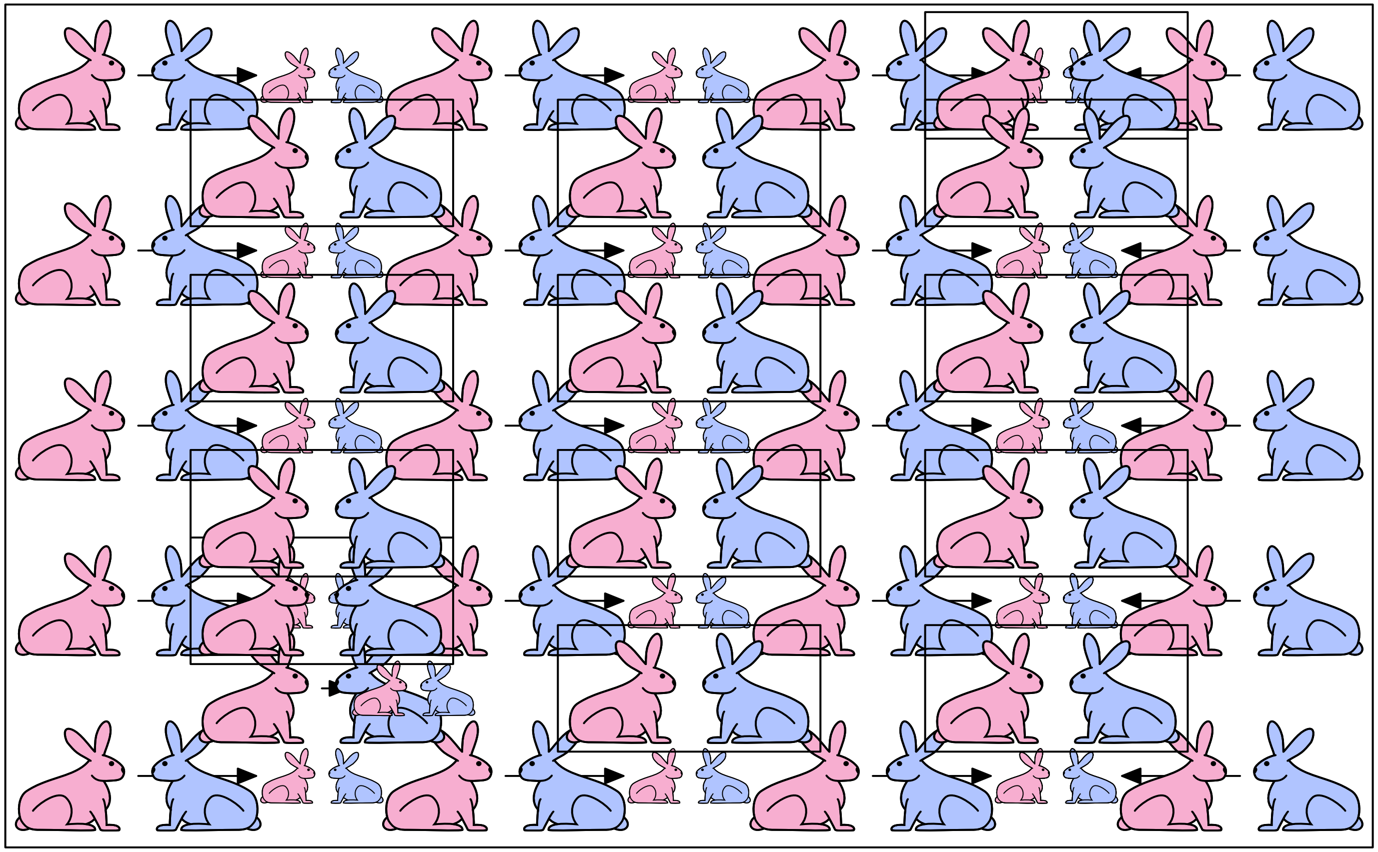

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FIBONACCI

After one month (call it 0) – 1 female

After second month – still 1 female (now pregnant)

After third month – two females, one pregnant, one not

In general, females(n) = females(n-1) + females(n-2)

- Every female alive at month n-2 will produce one female in month n;
- These can be added those alive in month n-1 to get total alive in month n

Month	Females
0	1

FIBONACCI

- Base cases:
  - Females(0) = 1
  - Females(1) = 1
- Recursive case
  - Females(n) = Females(n-1) + Females(n-2)

FIBONACCI

```
def fib(x):
    """assumes x an int >= 0
       returns Fibonacci of x"""
    if x == 0 or x == 1:
        return 1
    else:
        return fib(x-1) + fib(x-2)
```

RECURSION ON NON-NUMERICS

- how to check if a string of characters is a palindrome, i.e., reads the same forwards and backwards
  - “Able was I, ere I saw Elba” – attributed to Napoleon
  - “Are we not drawn onward, we few, drawn onward to new era?” – attributed to Anne Michaels

Image courtesy of wikipedia, in the public domain.

SOLVING RECURSIVELY?

- First, convert the string to just characters, by stripping out punctuation, and converting upper case to lower case
- Then
  - Base case: a string of length 0 or 1 is a palindrome
  - Recursive case:
    - If first character matches last character, then is a palindrome if middle section is a palindrome

EXAMPLE

- ‘Able was I, ere I saw Elba’ → ‘ablewasiereisawleba’
- `isPalindrome('ablewasiereisawleba')`
  is same as
  - `'a' == 'a' and isPalindrome('blewasiereisawleb')`

```python
def isPalindrome(s):

    def toChars(s):
        s = s.lower()
        ans = ''
        for c in s:
            if c in 'abcdefghijklmnopqrstuvwxyz':
                ans = ans + c
        return ans

    def isPal(s):
        if len(s) <= 1:
            return True
        else:
            return s[0] == s[-1] and isPal(s[1:-1])

    return isPal(toChars(s))
```

DIVIDE AND CONQUER

- an example of a “divide and conquer” algorithm
- solve a hard problem by breaking it into a set of sub-problems such that:
  - sub-problems are easier to solve than the original
  - solutions of the sub-problems can be combined to solve the original

DICTIONARIES

HOW TO STORE STUDENT INFO

- so far, can store using separate lists for every info

```
names = ['Ana', 'John', 'Denise', 'Katy']
grade = ['B', 'A+', 'A', 'A']
course = [2.00, 6.0001, 20.002, 9.01]
```

- a **separate list** for each item
- each list must have the **same length**
- info stored across lists at **same index**, each index refers to info for a different person

HOW TO UPDATE/RETRIEVE STUDENT INFO

```
def get_grade(student, name_list, grade_list, course_list):
    i = name_list.index(student)
    grade = grade_list[i]
    course = course_list[i]
    return (course, grade)
```

- **messy** if have a lot of different info to keep track of
- must maintain **many lists** and pass them as arguments
- must **always index** using integers
- must remember to change multiple lists

A BETTER AND CLEANER WAY – A DICTIONARY

- nice to **index item of interest directly** (not always int)
- nice to use **one data structure**, no separate lists

A list

0	Elem 1
1	Elem 2
2	Elem 3
3	Elem 4
…	…

index   element

A dictionary

Key 1	Val 1
Key 2	Val 2
Key 3	Val 3
Key 4	Val 4
…	…

custom index by label   element

A PYTHON DICTIONARY

- store pairs of data
  - key
  - value

'Ana'	'B'
'Denise'	'A'
'John'	'A+'
'Katy'	'A'

custom index by label

element

```
my_dict = {}
```

empty dictionary

```
grades = {'Ana':'B', 'John':'A+', 'Denise':'A', 'Katy':'A'}
```

key1 val1 key2 val2 key3 val3 key4 val4

DICTIONARY LOOKUP

- similar to indexing into a list
- **looks up** the **key**
- **returns** the **value** associated with the key
- if key isn't found, get an error

'Ana'	'B'
'Denise'	'A'
'John'	'A+'
'Katy'	'A'

```
grades = {'Ana':'B', 'John':'A+', 'Denise':'A', 'Katy':'A'}
grades['John']        → evaluates to 'A+'
grades['Sylvan']      → gives a KeyError
```

DICTIONARY OPERATIONS

'Ana'	'B'
'Denise'	'A'
'John'	'A+'
'Katy'	'A'
'Sylvan'	'A'

```
grades = {'Ana':'B', 'John':'A+', 'Denise':'A', 'Katy':'A'}
```

- **add** an entry

```
grades['Sylvan'] = 'A'
```

- **test** if key in dictionary

```
'John' in grades         → returns True
'Daniel' in grades       → returns False
```

- **delete** entry

```
del(grades['Ana'])
```

DICTIONARY OPERATIONS

'Ana'	'B'
'Denise'	'A'
'John'	'A+'
'Katy'	'A'

```
grades = {'Ana':'B', 'John':'A+', 'Denise':'A', 'Katy':'A'}
```

- get an **iterable that acts like a tuple of all keys** no guaranteed order

  `grades.keys()` → returns `['Denise','Katy','John','Ana']`

- get an **iterable that acts like a tuple of all values**

  `grades.values()` → returns `['A', 'A', 'A+', 'B']`

  no guaranteed order

DICTIONARY KEYS and VALUES

- values
  - any type (**immutable and mutable**)
  - can be **duplicates**
  - dictionary values can be lists, even other dictionaries!
- keys
  - must be **unique**
  - **immutable** type (`int`, `float`, `string`, `tuple`, `bool`)
    - actually need an object that is **hashable,** but think of as immutable as all immutable types are hashable
  - careful with `float` type as a key
- **no order** to keys or values!

```
d = {4:{1:0}, (1,3):"twelve", 'const':[3.14,2.7,8.44]}
```

`list` vs `dict`

- **ordered** sequence of elements
- look up elements by an integer index
- indices have an **order**
- index is an **integer**

- **matches** “keys” to “values”
- look up one item by another item
- **no order** is guaranteed
- key can be any **immutable** type

EXAMPLE: 3 FUNCTIONS TO ANALYZE SONG LYRICS

1) create a **frequency dictionary** mapping `str:int`

2) find **word that occurs the most** and how many times

- use a list, in case there is more than one word
- return a tuple `(list,int)` for (words_list, highest_freq)

3) find the **words that occur at least X times**

- let user choose "at least X times", so allow as parameter
- return a list of tuples, each tuple is a `(list, int)` containing the list of words ordered by their frequency
- IDEA: From song dictionary, find most frequent word. Delete most common word. Repeat. It works because you are mutating the song dictionary.

CREATING A DICTIONARY

```
def lyrics_to_frequencies(lyrics):
    myDict = {}
    for word in lyrics:
        if word in myDict:
            myDict[word] += 1
        else:
            myDict[word] = 1
    return myDict
```

can iterate over list

can iterate over keys in dictionary

update value associated with key

USING THE DICTIONARY

```
def most_common_words(freqs):
    values = freqs.values()
    best = max(values)
    words = []
    for k in freqs:
        if freqs[k] == best:
            words.append(k)
    return (words, best)
```

this is an iterable, so can apply built-in function

can iterate over keys in dictionary

LEVERAGING DICTIONARY PROPERTIES

```
def words_often(freqs, minTimes):
    result = []
    done = False
    while not done:
        temp = most_common_words(freqs)
        if temp[1] >= minTimes:
            result.append(temp)
            for w in temp[0]:
                del(freqs[w])
        else:
            done = True
    return result

print(words_often(beatles, 5))
```

can directly mutate dictionary; makes it easier to iterate

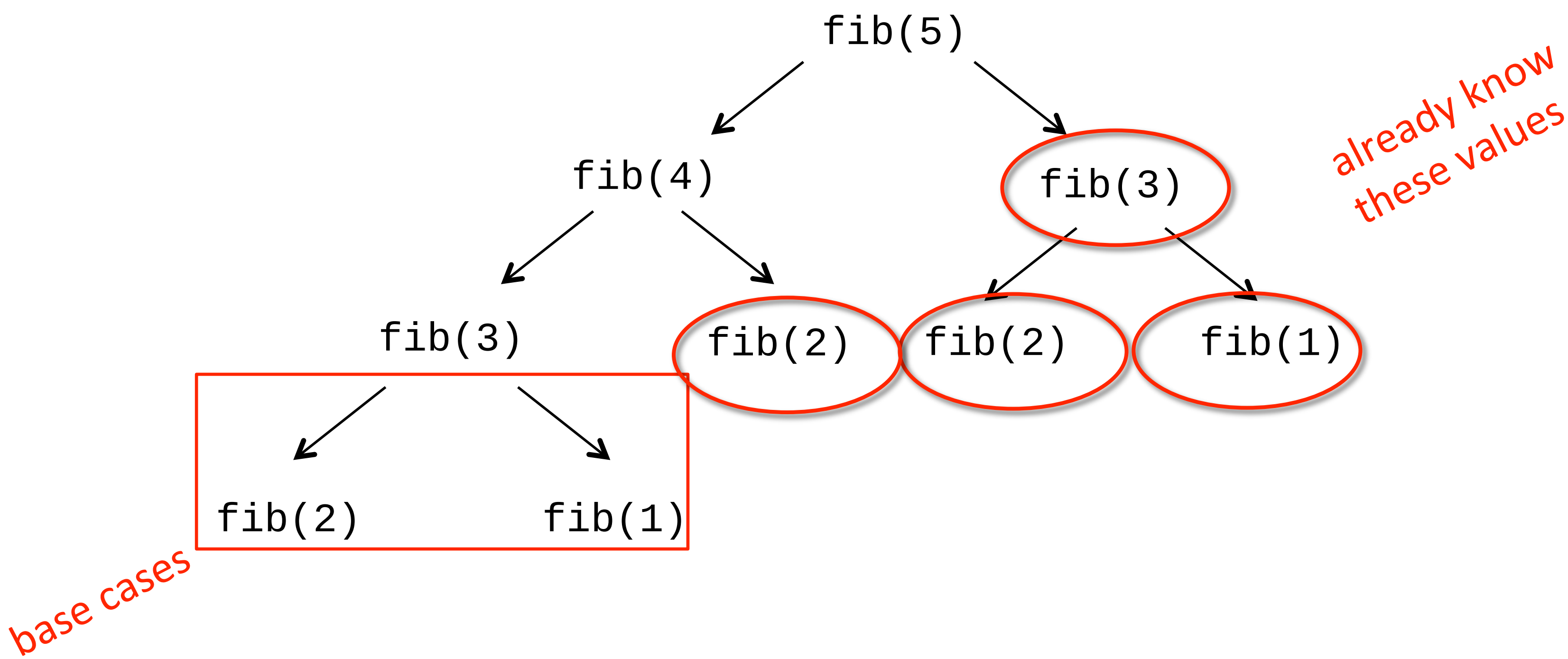
FIBONACCI RECURSIVE CODE

```
def fib(n):
    if n == 1:
        return 1
    elif n == 2:
        return 2
    else:
        return fib(n-1) + fib(n-2)
```

- two base cases
- calls itself twice
- this code is inefficient

INEFFICIENT FIBONACCI

`fib(n) = fib(n-1) + fib(n-2)`



- **recalculating** the same values many times!
- could keep **track** of already calculated values

FIBONACCI WITH A DICTIONARY

```
def fib_efficient(n, d):
    if n in d:
        return d[n]
    else:
        ans = fib_efficient(n-1, d) + fib_efficient(n-2, d)
        d[n] = ans
        return ans

d = {1:1, 2:2}
print(fib_efficient(6, d))
```

Method sometimes called " memoization"

Initialize dictionary with base cases

- do a **lookup first** in case already calculated the value
- **modify dictionary** as progress through function calls

EFFICIENCY GAINS

- Calling fib(34) results in 11,405,773 recursive calls to the procedure
- Calling fib_efficient(34) results in 65 recursive calls to the procedure
- Using dictionaries to capture intermediate results can be very efficient
- But note that this only works for procedures without side effects (i.e., the procedure will always produce the same result for a specific argument independent of any other computations between calls)