Solutions to Recitation: HO Model

1. Stolper-Samuelson Theorem

   (a) Firm's cost minimization problem is

   $$\min wL_i + rK_i$$
   $$st\ F_i \geq 1$$

   which has FOCs

   $$\begin{aligned} w &= \lambda(1-\beta_i)\left(\frac{L_i}{K_i}\right)^{-\beta_i} \\ r &= \lambda\beta_i\left(\frac{L_i}{K_i}\right)^{1-\beta_i} \\ &\Rightarrow \\ \frac{w}{r} &= \frac{1-\beta_i}{\beta_i}\left(\frac{K_i}{L_i}\right) \\ \frac{K_i}{L_i} &= \frac{a_{Ki}}{a_{Li}} = \frac{\beta_i}{1-\beta_i}\left(\frac{w}{r}\right) \end{aligned}$$

   (b) By definition

   $$a_{Li} = \frac{L_i}{Q_i} = \left(\frac{K_i}{L_i}\right)^{-\beta_i} = \left[\frac{\beta_i}{1-\beta_i}\left(\frac{w}{r}\right)\right]^{-\beta_i}$$

   where the last line follows from the result in a. Similarly

   $$a_{Ki} = \frac{K_i}{Q_i} = \left(\frac{K_i}{L_i}\right)^{1-\beta_i} = \left[\frac{\beta_i}{1-\beta_i}\left(\frac{w}{r}\right)\right]^{1-\beta_i}$$

   or this could have also been found by directly applying equation 4 to the expression for a_{Li}.

   (c) Firm's profit maximization implies

   $$\begin{aligned} p_i &= \frac{w}{MPL_i} = \frac{w}{(1-\beta_i)\left(\frac{K_i}{L_i}\right)^{\beta_i}} = \frac{w}{(1-\beta_i)\left(\frac{\beta_i}{1-\beta_i}\left(\frac{w}{r}\right)\right)^{\beta_i}} \\ &= \beta_i^{-\beta_i}(1-\beta_i)^{\beta_i-1} r^{\beta_i} w^{1-\beta_i} \end{aligned}$$

(d)

$$\frac{p_C}{p_F} = \frac{\beta_C^{-\beta_C}(1-\beta_C)^{\beta_C-1} r^{\beta_C} w^{1-\beta_C}}{\beta_F^{-\beta_F}(1-\beta_F)^{\beta_F-1} r^{\beta_F} w^{1-\beta_F}} = \frac{\beta_C^{-\beta_C}(1-\beta_C)^{\beta_C-1}}{\beta_F^{-\beta_F}(1-\beta_F)^{\beta_F-1}} \left(\frac{w}{r}\right)^{\beta_F-\beta_C}$$

which is increasing in w and decreasing in r. This implies the result.

2. Rybczynski Theorem

(a) Manipulating market clearing for Labor and applying the definition of a

$$\begin{aligned}
L_C &= L - L_F \\
Q_C a_{LC} &= L - Q_F a_{LF} \\
\frac{a_{LC}}{a_{KC}} K_C &= L - \frac{a_{LF}}{a_{KF}} K_F \\
\left(\frac{a_{LC}}{a_{KC}} - \frac{a_{LF}}{a_{KF}}\right) K_C &= L - \frac{a_{LF}}{a_{KF}} K \\
Q_C &= \frac{L - \frac{a_{LF}}{a_{KF}} K}{a_{KC}\left(\frac{a_{LC}}{a_{KC}} - \frac{a_{LF}}{a_{KF}}\right)}
\end{aligned}$$

Similarly for Food

$$\begin{aligned}
-L_F &= L_C - L \\
-\frac{a_{LF}}{a_{KF}} K_F &= \frac{a_{LC}}{a_{KC}} K_C - L \\
\left(\frac{a_{LC}}{a_{KC}} - \frac{a_{LF}}{a_{KF}}\right) K_F &= \frac{a_{LC}}{a_{KC}} K - L \\
Q_F &= \frac{\frac{a_{LC}}{a_{KC}} K - L}{a_{KF}\left(\frac{a_{LC}}{a_{KC}} - \frac{a_{LF}}{a_{KF}}\right)}
\end{aligned}$$

(b) Applying equation 4

$$\frac{a_{LC}}{a_{KC}} - \frac{a_{LF}}{a_{KF}} = \left(\frac{1-\beta_C}{\beta_C} - \frac{1-\beta_F}{\beta_F}\right)\frac{r}{w} = \left(\frac{\beta_F-\beta_C}{\beta_C\beta_F}\right)\frac{r}{w} > 0$$

All of the a's are positive by definition and fixed by the assumption of small open economy, so it is easy to see the result from the expressions in the previous part.

3. Heckscher-Ohlin Theorem

(a) This is just the standard result for Cobb-Douglas preferences that we have seen. Quickly solving for demand

$$\begin{aligned}(1-\alpha)\left(\frac{D_C}{D_F}\right)^{-\alpha} &= \lambda p_C \\ \alpha\left(\frac{D_C}{D_F}\right)^{1-\alpha} &= \lambda p_F \\ \frac{p_C}{p_F} &= \frac{1-\alpha}{\alpha}\frac{D_F}{D_C} \\ p_C D_C &= \frac{1-\alpha}{\alpha} p_F D_F\end{aligned}$$

summing both sides gives the result

(b) Plugging in from 1d for $\frac{p_C}{p_F}$ into the previous expression

$$\frac{\beta_C^{-\beta_C}\left(1-\beta_C\right)^{\beta_C-1}}{\beta_F^{-\beta_F}\left(1-\beta_F\right)^{\beta_F-1}}\left(\frac{w}{r}\right)^{\beta_F-\beta_C}\frac{Q_C}{Q_F}=\frac{1-\alpha}{\alpha}$$

(c)

$$\begin{aligned}\frac{Q_C}{Q_F} &= \frac{\frac{L-\frac{a_{LF}}{a_{KF}}K}{a_{KC}}}{\frac{\frac{a_{LC}}{a_{KC}}K-L}{a_{KF}}} = \frac{a_{KF}}{a_{KC}}\left(\frac{L-\frac{a_{LF}}{a_{KF}}K}{\frac{a_{LC}}{a_{KC}}K-L}\right) = \frac{a_{KF}}{a_{KC}}\left(\frac{\frac{L}{K}-\frac{a_{LF}}{a_{KF}}}{\frac{a_{LC}}{a_{KC}}-\frac{L}{K}}\right) \\ &= \left(\frac{\left(\frac{\beta_F}{1-\beta_F}\frac{w}{r}\right)^{1-\beta_F}}{\left(\frac{\beta_C}{1-\beta_C}\frac{w}{r}\right)^{1-\beta_C}}\right)\left(\frac{\frac{L}{K}-\frac{1-\beta_F}{\beta_F}\left(\frac{r}{w}\right)}{\frac{1-\beta_C}{\beta_C}\left(\frac{r}{w}\right)-\frac{L}{K}}\right) \\ &= \left(\frac{\beta_F}{1-\beta_F}\right)^{1-\beta_F}\left(\frac{\beta_C}{1-\beta_C}\right)^{\beta_C-1}\left(\frac{w}{r}\right)^{\beta_C-\beta_F}\left(\frac{\frac{L}{K}-\frac{1-\beta_F}{\beta_F}\left(\frac{r}{w}\right)}{\frac{1-\beta_C}{\beta_C}\left(\frac{r}{w}\right)-\frac{L}{K}}\right)\end{aligned}$$

Then using the result in (b)

$$\begin{aligned}&\left(\frac{1-\alpha}{\alpha}\right)\frac{\beta_C^{\beta_C}\left(1-\beta_C\right)^{1-\beta_C}}{\beta_F^{\beta_F}\left(1-\beta_F\right)^{1-\beta_F}}\left(\frac{w}{r}\right)^{\beta_C-\beta_F} \\ = &\left(\frac{\beta_F}{1-\beta_F}\right)^{1-\beta_F}\left(\frac{\beta_C}{1-\beta_C}\right)^{\beta_C-1}\left(\frac{w}{r}\right)^{\beta_C-\beta_F}\left(\frac{\frac{L}{K}-\frac{1-\beta_F}{\beta_F}\left(\frac{r}{w}\right)}{\frac{1-\beta_C}{\beta_C}\left(\frac{r}{w}\right)-\frac{L}{K}}\right)\end{aligned}$$

and

$$\left(\frac{1-\alpha}{\alpha}\right)\frac{\beta_C^{\beta_C}\beta_C^{1-\beta_C}}{\beta_F^{\beta_F}\beta_F^{1-\beta_F}}=\frac{\left(\frac{L}{K}\right)\left(\frac{w}{r}\right)-\frac{1-\beta_F}{\beta_F}}{\frac{1-\beta_C}{\beta_C}-\left(\frac{L}{K}\right)\left(\frac{w}{r}\right)}$$

The LHS is a constant while the RHS is increasing in $\frac{L}{K}$ and $\frac{w}{r}$. Hence a country with a higher $\frac{L}{K}$ will have a lower $\frac{w}{r}$. From the previous expression for relative price, a lower $\frac{w}{r}$ implies a lower $\frac{p_C}{p_F}$, which in turn implies a higher $\frac{Q_C}{Q_F}$. The law of comparative advantage states that the country with the lower $\frac{p_C}{p_F}$ will export Cloth, which all else equal will be the country with the higher $\frac{L}{K}$ by the previous analysis.