Chapter 4

Applications/Variations

4.1 Consumption Smoothing

4.1.1 The Intertemporal Budget

- For any given sequence of interest rates $\{R_t\}_{t=0}^{\infty}$, pick an arbitrary $q_0 > 0$ and define $\{q_t\}_{t=1}^{\infty}$, by

$$q_t = \frac{q_0}{(1+R_1)...(1+R_t)}.$$

q_t represents the price of period$-t$ consumption relative to period-0 consumption.

- The budget in period t is given by

$$c_t + a_{t+1} \leq (1+R_t)a_t + y_t$$

where a_t denotes assets and y_t denotes labor income.

- Multiplying the period-t budget by q_t, adding up over all t, and using the fact that $q_t(1+R_t) = q_{t-1}$, we have

$$\sum_{t=0}^{T} q_t c_t + q_T a_{T+1} \leq q_0(1+R_0)a_0 + \sum_{t=1}^{T} q_t y_t$$

- Assuming either that the agent dies at finite time without leaving any bequests, in which case $a_{T+1} = 0$, or that the time is infinite, in which case we impose $q_T a_{T+1} \to 0$ as $T \to \infty$, we conclude that the intertemporal budget constraint is given by

$$\sum_{t=0}^{T} q_t c_t \leq q_0(1+R_0)a_0 + \sum_{t=1}^{T} q_t y_t,$$

  whether $T < \infty$ (finite horizon) or $T = \infty$ (infinite horizon).

- The interpretation is simple: The present value of the consumption the agent enjoys from period 0 and on can not exceed the value of initial assets in period 0 plus the present value of the labor income received from period 0 and on.

- We can rewrite the intertemporal budget as

$$\sum_{t=0}^{T} q_t c_t \leq q_0 x_0$$

where

$$x_0 \equiv (1+R_0)a_0 + h_0 \qquad \text{and} \qquad h_0 \equiv \sum_{t=0}^{\infty} \frac{q_t}{q_0} y_t.$$

- $(1+R_0)a_0$ is the household's *financial wealth* as of period 0.

- h_0 is the present value of labor income as of period 0; we often call h_0 the household's *human wealth* as of period 0.

- The sum $x_0 \equiv (1+R_0)a_0 + h_0$ represents the household's *effective wealth*.

- Note that the sequence of per-period budgets and the intertemporal budget constraint are equivalent. We can thus rewrite the household's consumption problem as follows

$$\max \sum_{t=0}^{T} \beta^t U(c_t)$$
$$s.t. \sum_{t=0}^{T} q_t c_t \leq q_0 x_0$$

- The above is like a "static" consumption problem: interpret c_t as different consumption goods and q_t as the price of these goods. This observation relates to the context of Arrow-Debreu markets that we discuss later.

4.1.2 Consumption Smoothing

- The Lagrangian for the household's problem is

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t U(c_t) + \lambda \left[q_0 x_0 - \sum_{t=0}^{T} q_t c_t \right]$$

  where λ is the shadow cost of resources for the consumer (that is, the Lagrange multiplier for the intertemporal budget constraint).

- The FOCs give

$$U'(c_0) = \lambda q_0$$

  for period 0 and similarly

$$\beta^t U'(c_t) = \lambda q_t$$

  for any period t.

- Suppose the interest rate equals the discount rate in all t: $R_t = \rho \equiv 1/\beta - 1$ and hence

$$q_t = \beta^t q_0$$

  The FOCs then reduce to

$$U'(c_t) = \lambda q_0 \; \forall t$$

  It follows that consumption is constant over time: there exists a $\overline{c}$ such that $c_t = \overline{c}$ for all t.

- But how is the value of $\overline{c}$ determined? From the intertemporal budget, using $q_t = \beta^t q_0$ and $c_t = \overline{c}$, we infer

$$q_0 x_0 = \sum_{t=0}^{T} q_t c_t = \frac{1}{1-\beta} q_0 \overline{c}$$

  and therefore the household consumes a constant fraction $1-\beta$ of his initial effective wealth in every period:

$$\overline{c} = (1-\beta) x_0 = (1-\beta) \left[(1+R_0) a_0 + h_0 \right]$$

4.2 Arrow-Debreu Markets

- We now introduce uncertainty. Each period t nature draws a random variable s_t, which has (for simplicity) finite support $\mathcal{S}$. Let $s^t = \{s_0, s_1, ..., s_t\}$ denote the entire history of these realization up to (and including) period t.

- At time 0, we open a complete set of Arrow-Debreu markets: agents can trade commodities contingent on all possible realizations of future uncertainty. If there are T period and S possible states per period, this means that we open $1 + S + S^2 + S^3 + ... + S^T$ markets.

- Let $q(s^t)$ be the period-0 price of a unit of the consumption good in period t and event s^t, and $w(s^t)$ the corresponding wage rate in terms of period-t, event-s^t consumption goods. $q(s^t)w(s^t)$ is then the period-t, event-s^t wage rate in terms of period-0 consumption goods.

- We can then write household's consumption problem as follows

$$\max \sum_t \sum_{s^t} \beta^t \pi(s^t) U\left(c^j(s^t), z^j(s^t)\right)$$

$$s.t. \quad \sum_t \sum_{s^t} [q(s^t) \cdot c^j(s^t) + q(s^t) w(s^t) \cdot z^j(s^t)] \;\leq\; q_0 \cdot \bar{x}_0^j$$

where

$$\bar{x}_0^j \equiv (1 + R_0) a_0 + \bar{h}_0^j,$$

$$\bar{h}_0^j \equiv \sum_{t=0}^{\infty} \frac{q(s^t)}{q_0} [w(s^t) \cdot \bar{z} - T^j(s^t)].$$

$(1+R_0)a_0^j$ is the household's *financial wealth* as of period 0; $\bar{z}$ is the endowment of time (normalize it to 1, if you like); $T^j(s^t)$ is a lump-sum tax obligation, which may depend on the identity of household but not on its choices. $\bar{h}_0^j$ is the present value of the entire time endowment as of period 0 net of taxes. (caution: this h is different than the one previous defined.)

4.2.1 The Consumption Problem with CEIS

- Suppose for a moment that preferences are separable between consumption and leisure and are homothetic with respect to consumption:

$$U(c,z) = u(c) + v(z) \qquad u(c) = \frac{c^{1-1/\theta}}{1-1/\theta}$$

- Letting μ be the Lagrange multiplier for the intertemporal budget constraint, the FOCs imply

$$\beta^t \pi(s^t) u'\left(c^j(s^t)\right) = \mu q(s^t)$$

for all $t \geq 0$. Evaluating this at $t = 0$, we infer $\mu = u'(c_0^j)$. It follows that the price of a consumable in period t relative to period 0 equals the marginal rate of intertemporal substitution between 0 and t :

$$\frac{q(s^t)}{q_0} = \frac{\beta^t \pi(s^t) u'\left(c^j(s^t)\right)}{u'(c_0^j)} = \beta^t \pi(s^t) \left(\frac{c^j(s^t)}{c_0^j}\right)^{-1/\theta}.$$

- Solving the latter for $c^j(s^t)$ gives $c^j(s^t) = c_0^j \left[\beta^t \pi(s^t)\right]^\theta \left[\frac{q(s^t)}{q_0}\right]^{-\theta}$. It follows that the present value of consumption is given by

$$\sum_t \sum_{s^t} q(s^t) c^j(s^t) = q_0^{-\theta} c_0^j \sum_{t=0}^{\infty} \left[\beta^t \pi(s^t)\right]^\theta q(s^t)^{1-\theta}$$

Substituting into the resource constraint, and solving for c_0, we conclude

$$c_0^j = m_0 \cdot x_0^j$$

where m_0 represents the MPC out of effective wealth as of period 0 and is given by

$$m_0 \equiv \frac{1}{\sum_{t=0}^{\infty} \left[\beta^t \pi(s^t)\right]^\theta \left[q(s^t)/q_0\right]^{1-\theta}}.$$

- A similar result applies for all $t \geq 0$. We conclude

Proposition 19 *Suppose preferences are separable between consumption and leisure and homothetic in consumption (CEIS). Then, the optimal consumption is linear in contemporaneous effective wealth:*

$$c_t^j = m_t \cdot x_t^j$$

where

$$x_t^j \equiv (1 + R_t)a_t^j + h_t^j,$$
$$h_t^j \equiv \sum_{\tau=t}^{\infty} \frac{q_t}{q_t}[w_\tau l_\tau^j - T_\tau^j],$$
$$m_t \equiv \frac{1}{\sum_{\tau=t}^{\infty} \beta^{\theta(\tau-t)}(q_\tau/q_t)^{1-\theta}}.$$

m_t is a decreasing (increasing) function of q_τ for any $\tau \geq t$ if and only $\theta > 1$ ($\theta < 1$). That is, the marginal propensity to save out of effective wealth is increasing (decreasing) in future interest rates if and only if the elasticity of intertemporal substitution is higher (lower) than unit. Moreover, for given prices, the optimal consumption path is independent of the timining of either labor income or taxes.

- Obviously, a similar result holds with uncertainty, as long as there are complete Arrow-Debreu markets.

- Note that any expected change in income has no effect on consumption as long as it does not affect the present value of labor income. Also, if there is an innovation (unexpected change) in income, consumption will increase today and for ever by an amount proportional to the innovation in the annuity value of labor income.

- To see this more clearly, suppose that the interest rate is constant and equal to the discount rate: $R_t = 1/\beta - 1$ for all t. Then, $c_t^j = c_0^j$ for all t and the consumer chooses a totally flat consumption path, no matter what is the time variation in labor income. And any unexpected change in consumption leads to a parallel shift in the path of consumption by an amount equal to the annuity value of the change in labor income. This is the manifestation of *intertemporal consumption smoothing.*

4.2.2 Incomplete Markets and Self-Insurance

- The above analysis has assumed no uncertainty, or that markets are complete. Extending the model to introduce idiosyncratic uncertainty in labor income would imply an Euler condition of the form

$$u'(c_t^j) = \beta(1+R)\mathbb{E}_t u'(c_{t+1}^j)$$

Note that, because of the convexity of u', as long as $Var_t[c_{t+1}^j] > 0$, we have $\mathbb{E}_t u'(c_{t+1}^j) > u'(\mathbb{E}_t c_{t+1}^j)$ and therefore

$$\frac{\mathbb{E}_t c_{t+1}^j}{c_t^j} > [\beta(1+R)]^\theta$$

This extra kick in consumption growth reflects the *precautionary motive for savings.* It remains true that transitory innovations in income result to persistent changes in consumption (because of consumption smoothing). At the same time, consumers find it optimal to accumulate a *buffer stock,* as a vehicle for self-insurance.

4.3 Aggregation and the Representative Consumer

- Consider a deterministic economy populated by many heterogeneous households. Households differ in their initial asset positions and (perhaps) their streams of labor income, but not in their preferences. They all have CEIS preferences, with identical θ.

- Following the analysis of the previous section, consumption for individual j is given by

$$c_t^j = m_t \cdot x_t^j.$$

Note that individuals share the same MPC out of effective wealth because they have identical θ.

- Adding up across households, we infer that aggregate consumption is given by

$$c_t = m_t \cdot x_t$$

where

$$x_t \equiv (1 + R_t)a_t + h_t,$$

$$h_t \equiv \sum_{\tau=t}^{\infty} \frac{q_\tau}{q_t}[w_\tau l_\tau - T_\tau],$$

$$m_t \equiv \frac{1}{\sum_{\tau=t}^{\infty} \beta^{\theta(\tau-t)}(q_\tau/q_t)^{1-\theta}}.$$

- Next, recall that individual consumption growth satisfies

$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t^j)}{u'(c_0^j)} = \beta^t \left(\frac{c_t^j}{c_0^j}\right)^{-1/\theta},$$

for every j. But if all agents share the same consumption growth rate, this should be the aggregate one. Therefore, equilibrium prices and aggregate consumption growth satisfy

$$\frac{q_t}{q_0} = \beta^t \left(\frac{c_t}{c_0}\right)^{-1/\theta}$$

Equivalently,

$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t)}{u'(c_0)}.$$

- Consider now an economy that has a single consumer, who is endowed with wealth x_t and has preferences

$$U(c) = \frac{c^{1-1/\theta}}{1-1/\theta}$$

The Euler condition for this consumer will be

$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t)}{u'(c_0)}.$$

Moreover, this consumer will find it optimal to choose consumption

$$c_t = m_t \cdot x_t.$$

But these are exactly the aggregative conditions we found in the economy with many agents.

- That is, the two economies share exactly the same equilibrium prices and allocations. It is in this sense that we can think of the single agent of the second economy as the "representative" agent of the first multi-agent economy.

- Note that here we got a stronger result than just the existence of a representative agent. Not only a representative agent existed, but he also had exactly the same preferences as each of the agents of the economy. This was true only because agents had identical and homothetic preferences. With other Gorman-type preferences, the preferences of the representative agent are "weighted average" of the population preferences, with the weights depending on the wealth distribution.

- Finally, note that these aggregation results extend easily to the case of uncertainty as long as markets are complete.

4.4 Fiscal Policy

4.4.1 Ricardian Equilivalence

- The intertemporal budget for the representative household is given by

$$\sum_{t=0}^{\infty} q_t c_t \leq q_0 x_0$$

where

$$x_0 = (1+R_0)a_0 + \sum_{t=0}^{\infty} \frac{q_t}{q_0}[w_t l_t - T_t]$$

and $a_0 = k_0 + b_0$.

- On the other hand, the intertemporal budget constraint for the government is

$$\sum_{t=0}^{\infty} q_t g_t + q_0(1+R_0)b_0 = \sum_{t=0}^{\infty} q_t T_t$$

- Substituting the above into the formula for x_0, we infer

$$x_0 = (1 + R_0)k_0 + \sum_{t=0}^{\infty} \frac{q_t}{q_0} w_t l_t - \sum_{t=0}^{\infty} \frac{q_t}{q_0} g_t$$

That is, aggregate household wealth is independent of either the outstanding level of public debt or the timing of taxes.

- We can thus rewrite the representative household's intertemporal budget as

$$\sum_{t=0}^{\infty} q_t [c_t + g_t] \leq q_0 (1 + R_0) k_0 + \sum_{t=0}^{\infty} q_t w_t l_t$$

Since the representative agent's budget constraint is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$, his consumption and labor supply will also be independent. But then the resource constraint implies that aggregate investment will be unaffected as well. Therefore, the aggregate path $\{c_t, k_t\}_{t=0}^{\infty}$ is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$. All that matter is the stream of government spending, not the way this is financed.

- More generally, consider now arbitrary preferences and endogenous labor supply, but suppose that the tax burden and public debt is uniformly distributed across households. Then, for *every* individual j, effective wealth is independent of either the level of public debt or the timing of taxes:

$$x_0^j = (1+R_0)k_0^j + \sum_{t=0}^{\infty} \frac{q_t}{q_0} w_t l_t^j - \sum_{t=0}^{\infty} \frac{q_t}{q_0} g_t,$$

  Since the individual's intertemporal budget is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$, her optimal plan $\{c_t^j, l_t^j, a_t^j\}_{t=0}^{\infty}$ will also be independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$ for any given price path. But if individual behavior does not change for given prices, markets will continue to clear for the same prices. That is, equilibrium prices are indeed also independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$. We conclude

Proposition 20 *Equilibrium prices and allocations are independent of either the intial level of public debt, or the mixture of deficits and (lump-sum) taxes that the government uses to finance governement spending.*

- *Remark:* For Ricardian equivalence to hold, it is critical both that markets are complete (so that agents can freely trade the riskless bond) and that horizons are infinite (so that the present value of taxes the household expects to pay just equals the amount of public debt it holds). If either condition fails, such as in OLG economies or economies with borrowing constraints, Ricardian equivalence will also fail. Ricardian equivalence may also fail if there are

4.4.2 Tax Smoothing and Debt Management

topic covered in class

notes to be completed

4.5 Asset Pricing and CCAPM

- Consider an asset that costs 1 unit in period t and delivers $1+\widetilde{r}_{t+1}$ in period $t+1$, where $\widetilde{r}_{t+1} = \widetilde{r}(s^{t+1})$ is possibly random. By arbitrage,

$$q(s^t) = \sum_{s^{t+1}|s^t} q(s^{t+1}) \left[1 + \widetilde{r}(s^{t+1})\right]$$

- Using

$$\mu q(s^t) = \beta^t \pi(s^t) u' \left(c(s^t)\right)$$

we infer

$$u'\left(c\left(s^t\right)\right) = \beta \sum_{s^{t+1}|s^t} \Pr\left[s^{t+1}|s^t\right] \left[1 + \widetilde{r}(s^{t+1})\right] u'\left(c\left(s^{t+1}\right)\right)$$

or equivalently

$$u'\left(c_t\right) = \beta \mathbb{E}_t \left[\left(1 + \widetilde{r}_{t+1}\right) u'\left(c_{t+1}\right)\right]$$

- It follows that

$$u'(c_t) = \beta\left\{\mathbb{E}_t\left[1+\widetilde{r}_{t+1}\right]\mathbb{E}_t\left[u'(c_{t+1})\right] + Cov_t\left[1+\widetilde{r}_{t+1}, u'(c_{t+1})\right]\right\}$$

$$\begin{aligned}\frac{u'(c_t)}{\beta\mathbb{E}_t\left[u'(c_{t+1})\right]} &= \mathbb{E}_t\left[1+\widetilde{r}_{t+1}\right] + \frac{Cov_t\left[1+\widetilde{r}_{t+1}, u'(c_{t+1})\right]}{\mathbb{E}_t\left[u'(c_{t+1})\right]} \\ &\approx 1+\mathbb{E}_t\left[\widetilde{r}_{t+1}\right] - \gamma Cov_t\left[\widetilde{r}_{t+1}, \frac{c_{t+1}}{c_t}\right]\end{aligned}$$

- For the riskless bond

$$\frac{u'(c_t)}{\beta\mathbb{E}_t\left[u'(c_{t+1})\right]} = 1 + r^*_{t+1}$$

- It follows that

$$\mathbb{E}_t\left[\widetilde{r}_{t+1}\right] - r^*_{t+1} = \gamma Cov_t\left[\widetilde{r}_{t+1}, \frac{c_{t+1}}{c_t}\right]$$

- For the US data 1980-1979,

$$
\begin{aligned}
Var\left[\frac{c_{t+1}}{c_t}\right]^{1/2} &= 3.6\% \\
Var\left[\widetilde{r}_{t+1}\right]^{1/2} &= 16.7\% \\
Corr\left[\widetilde{r}_{t+1}, \frac{c_{t+1}}{c_t}\right] &= 40\% \\
Cov\left[\widetilde{r}_{t+1}, \frac{c_{t+1}}{c_t}\right] &= 0.24\% \\
\mathbb{E}_t\left[\widetilde{r}_{t+1}\right] - r^*_{t+1} &= 6\%
\end{aligned}
$$

To match these data, one needs

$$\gamma = 25$$