Lecture Note 3 *

Multiple Random Variables (Multivariate Model)

MIT 14.30 Spring 2006

Herman Bennett

6 Multiple Random Variables

6.1 Bivariate Distribution

Many experiments deal with more than one source of uncertainty. For these cases a **random vector** must be defined to contain the multiple random variables we are interested in.

An n-dimensional random vector is a function from a sample space S into $\Re^n$. In the bivariate case, $n = 2$.

6.1.1 Discrete Model

Let (X, Y) be a discrete bivariate random vector. The joint pmf of the random vector (X, Y) is the function $f_{XY}(x, y)$, defined by:

$$f_{XY}(x, y) = P(X = x, Y = y) \quad \text{for all } x \text{ and } y; \tag{9}$$

and satisfies the following properties:

$$\begin{array}{ll} i) & f(x, y) \geq 0 \quad \text{for all pairs } (x, y). \\ ii) & \sum_{\forall x} \sum_{\forall y} f(x, y) = 1. \end{array}$$

- Note that $f_{XY}(x, y) : \Re^2 \to \Re$.

*Caution: These notes are not necessarily self-explanatory notes. They are to be used as a complement to (and not as a substitute for) the lectures.

• As in the univariate case, in the bivariate case an event A is defined as a subcollection of outcomes (x, y). The probability of event A is given by:

$$P((X,Y) \in A) = \sum_{(x,y)\in A} f(x,y). \tag{10}$$

• As in the univariate case, the bivariate distribution of (X, Y) can be completely characterized by its joint pmf or its joint cdf. The joint cdf of (X, Y) is the function $F(x, y)$, defined by:

$$F(x,y) = P(X \leq x, Y \leq y) = \sum_{X\leq x} \sum_{Y\leq y} f(x,y). \tag{11}$$

Example 6.1. Check the properties of $f(x, y)$, and compute $P(X \geq 2, Y \geq 3)$, $P(X = 2)$, and $P(|X - Y| = 1)$.

$f(x,y)$	0	1	2	3	4
0	.1	.05	.05	0	0
1	.05	.2	.2	.05	0
2	0	0	.1	.1	.05
3	0	0	0	0	.05

6.1.2 Continuous Model

Let (X, Y) be a continuous bivariate random vector. The joint pdf of (X, Y) is the function $f_{XY}(x, y)$, defined by:

$$P((X,Y) \in A) = \int_A \int f_{XY}(x,y)dxdy, \qquad \text{for every subset } A \text{ of the } xy\text{-plane}; \tag{12}$$

and satisfies the following properties:

$$\begin{aligned} &i) \quad f_{XY}(x,y) \geq 0 \quad \text{for all } (x,y). \\ &ii) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1. \end{aligned}$$

- Note again that $f_{XY}(x, y) : \Re^2 \rightarrow \Re$.

- As in the continuous univariate case, the bivariate distribution of (X, Y) can be completely characterized by its joint pdf or its joint cdf. The joint cdf of (X, Y) is the function $F(x, y)$, defined by:

$$F(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)dudv. \tag{13}$$

- $F(-\infty, y) = 0, \qquad F(x, -\infty) = 0, \qquad F(\infty, \infty) = 1.$

- If (X, Y) is a continuous random vector, then $P(X = x_0, Y = y_0) = ?$

- $\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x, y)$ at continuous points of $f(x, y)$.

Example 6.2. Check the properties of $f(x, y)$ and compute $P(X \leq 0.6, Y \leq 0.6)$ and $P(X + Y > 1)$.

$$f(x,y) = \begin{cases} 6xy^2 & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

6.2 Marginal Distribution

We use the concept of marginal distribution to illustrate the fact that from a bivariate distribution it is possible to recover the univariate distribution of each of the random variables included in the random vector.

Let (X, Y) be a random vector with joint pmf/pdf $f_{XY}(x, y)$. The marginal pmf's/pdf's of X and Y are the functions $f_X(x)$ and $f_Y(y)$, defined by:

$$f_X(x) = \sum_{y \in \Re} f_{XY}(x, y) \qquad \text{and} \qquad f_Y(y) = \sum_{x \in \Re} f_{XY}(x, y) \quad \text{(discrete model)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx \quad \text{(continuous model)} \tag{14}$$

- As with any pmf/pdf, $f_X(x)$ and $f_Y(y)$ must satisfy $i)$ $f() \geq 0$ and $ii)$ $\sum / \int = 1$.

- It is not always possible to recover the joint distribution of (X, Y) from the marginal distributions, $f_X(x)$ and $f_Y(y)$, because the marginal distributions do not contain the information about the relationship between the variables (unless they are independent, more on this later).

Example 6.3. Following Example 6.1, find $f_X(x)$ and $f_Y(y)$.

Example 6.4. Following Example 6.2, find $f_X(x)$ and $f_Y(y)$.

6.3 Conditional Distribution

Let (X, Y) be a random vector with joint pmf/pdf $f_{XY}(x, y)$ and marginal pmf's/pdf's $f_X(x)$ and $f_Y(y)$. For any x such that $f_X(x) > 0$, the conditional pmf/pdf of Y given $X = x$, denoted $f(y|x)$, is given by:

$$f(y|x) = P(Y = y|X = x) = \frac{f_{XY}(x, y)}{f_X(x)} \qquad \text{and} \qquad f(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} \tag{15}$$

(discrete model) (continuous model)

For any y such that $f_Y(y) > 0$, the conditional pmf/pdf of X given $Y = y$, denoted $f(x|y)$, is given by:

$$f(x|y) = P(X = x|Y = y) = \frac{f_{XY}(x, y)}{f_Y(y)} \qquad \text{and} \qquad f(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} \tag{16}$$

(discrete model) (continuous model)

- As with any pmf/pdf, $f(x|y)$ and $f(y|x)$ must satisfy *i)* $f() \geq 0$ and *ii)* $\sum / \int = 1$.
- Intuition:

– Knowing the value of the RV X implies that many outcomes (x, y), that before knowing X were possible, are now impossible outcomes (zero mass).

– As a result, the property $\sum / \int = 1$ is not satisfied anymore and we need to rescale the probabilities of the still-possible outcomes (dividing the joint by the marginal) in order to satisfy this property, while at the same time keeping constant the relative likelihood between the still-possible outcomes.

Example 6.5. Following Example 6.1, find $f(y|x = 1)$.

Example 6.6. Following Example 6.2, find $f(y|x = 0.5)$.

6.4 Independence

Let (X, Y) be a random vector with joint pmf/pdf $f(x, y)$ and marginal pmfs/pdfs $f_X(x)$ and $f_Y(y)$. RV X and RV Y are called independent random variables if:

$$f(x, y) = f_X(x)f_Y(y), \quad \text{for all } x \text{ and } y. \tag{17}$$

• Note that this implies: $f(y|x) = f_Y(y)$. The knowledge that $X = x$ gives no additional information about Y.

• $f_{X,Y}(x, y) = f_X(x)f_Y(y) \longleftrightarrow \ P(x, y) = P(x)P(y)$ (discrete model).

• A useful way to check independence: X and Y are independent $\longleftrightarrow \ f(x, y) = g(x)h(y)$ for all x and y.

Example 6.7. Following Example 6.1, are X and Y independent?

Example 6.8. Following Example 6.2, are X and Y independent?

6.5 Wrap-up

Example 6.9. $f(x,y) = \begin{cases} 8yx & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

i) Check that $f(x,y)$ satisfies the properties of a joint pdf.
ii) Find the marginal distribution of X and Y.
iii) Find $f(y|x = 0.5)$.
iv) Are X and Y independent?

Example 6.10. $f(x,y) = \begin{cases} cyx^2 & \text{if } x^2 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

i) Find c.
ii) Find the marginal distribution of X and Y.
iii) Find $f(y|x = 0.5)$.
iv) Are X and Y independent?

6.6 Multivariate Distribution

See attached handout.