0 (Four) Most Common Hypothesis Tests: An Applied Review

We review the four most common hypothesis test structures applying them to the same random sample example. We change the hypotheses and the decision rule in each test.

0.1 Random Sample

Let $X_1, ..., X_{10}$ be a random sample from a normal population with unknown mean (μ) and known standard deviation ($\sigma = 1$). The following table presents the realization of the random sample.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	$\bar{x}$
1.0	0.4	-0.3	1.2	-0.6	1.6	-0.3	2.0	0.5	-0.1	0.54

0.2 Likelihood Ratio Test (LRT):

Using a LR Test of size 5%, test the null hypothesis that the population mean is 0 against the alternative hypothesis that the population mean is 1.

$$H_0: \quad \mu = 0$$
$$H_1: \quad \mu = 1$$

Decision Rule form: "Reject H_0 if $f_1(\mathbf{x})/f_0(\mathbf{x}) > k$".

• We need to compute k and $f_i(\mathbf{x})$ for $i = 0, 1$. The value of k depends on the size of the test that we want to construct and the value of $f_i(\mathbf{x})$ depends on the realization of the random sample.

0.2.1 Computing k

To compute k we need to find the distribution of $f_1(\mathbf{X})/f_0(\mathbf{X})$ under the null hypothesis. Note that the capital letter for $\mathbf{X}$ represents the fact that the statistic $f_1(\mathbf{X})/f_0(\mathbf{X})$ is a random variable constructed as a function of the random variables in the random sample.

The way to compute the distribution of $f_1(\mathbf{X})/f_0(\mathbf{X})$ depends on the population distributions assumed (case specific) and many times it is very hard to compute. Once we know the distribution of $f_1(\mathbf{X})/f_0(\mathbf{X})$, we look for the value of k that satisfy the size-condition:

$$P\Big(f_1(\mathbf{X})/f_0(\mathbf{X}) > k|\mu = 0\Big) = \alpha = 0.05$$

DeGroot and Schervish (2002, pages 465-8) propose a strategy that allows us to compute k without necessarily knowing the distribution of $f_1(\mathbf{X})/f_0(\mathbf{X})$. Using this method, we find that $k = 1.22$, which implies that $P\big(f_1(\mathbf{X})/f_0(\mathbf{X}) > 1.22|\mu = 0\big) = 0.05$. I leave you to check the details in DeGroot and Schervish's book (not necessary for the exam).

0.2.2 Computing the likelihood ratio $f_1(\mathbf{x})/f_0(\mathbf{x})$

- Computing $f_0(\mathbf{x})$:

$$f_0(\mathbf{x}|\mu = 0) = \prod_{j=1}^{n} f_0(x_j|\mu = 0) = \prod_{j=1}^{10}\left(\frac{1}{\sqrt{2\pi}\sigma}\ e^{-(x-\mu)^2/(2\sigma^2)} \mid \mu = 0,\ \sigma = 1\right)$$
$$= \frac{1}{(2\pi)^{10/2}\cdot 1^{10}}\ e^{-(2\cdot 1^2)^{-1}\sum_{j=1}^{10}(x_j-0)^2} = \frac{1}{(2\pi)^{10/2}\cdot 1^{10}}\ e^{-(2\cdot 1^2)^{-1}9.96}$$

- Computing $f_1(\mathbf{x})$:

$$f_1(\mathbf{x}|\mu = 1) = \prod_{j=1}^{n} f_0(x_j|\mu = 1) = \prod_{j=1}^{10}\left(\frac{1}{\sqrt{2\pi}\sigma}\ e^{-(x-\mu)^2/(2\sigma^2)} \mid \mu = 1,\ \sigma = 1\right)$$
$$= \frac{1}{(2\pi)^{10/2}\cdot 1^{10}}\ e^{-(2\cdot 1^2)^{-1}\sum_{j=1}^{10}(x_j-1)^2} = \frac{1}{(2\pi)^{10/2}\cdot 1^{10}}\ e^{-(2\cdot 1^2)^{-1}9.16}$$

$$\Longrightarrow\ f_1(\mathbf{x})/f_0(\mathbf{x}) = e^{-0.5\cdot 9.16}/e^{-0.5\cdot 9.96} = e^{0.40} = 1.49$$

0.2.3 Result of the test

$f_1(\mathbf{x})/f_0(\mathbf{x}) = 1.49 > k = 1.22$, so we can reject at 5% the null hypothesis that the population mean is 0 against the alternative hypothesis that the population mean is 1.

0.3 One-sided Test:

Using a One-sided Test of size 6%, test the null hypothesis that the population mean is 0.4 against the alternative hypothesis that the population mean is higher than 0.4.

$$H_0: \quad \mu = 0.4$$
$$H_1: \quad \mu > 0.4$$

Decision Rule form: "Reject H_0 if $\bar{x} > c$".

• We need to compute c, which depends on the size of the test that we want to construct.

0.3.1 Computing c

We need to find the value of c that satisfies the condition that the probability of type 1 error is 6%.

$$P\Big(\bar{X} > c | \mu = 0.4\Big) = \alpha = 0.06$$

To compute c we need to find the distribution of the random variable $\bar{X}$. The random sample is normal, so we know that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. Under the null hypothesis $\bar{X} \sim N(0.4, \frac{1^2}{10})$, and therefore,

$$P\left(\frac{\sqrt{10}(\bar{X} - 0.4)}{1} > \frac{\sqrt{10}(c - 0.4)}{1}\right) = 0.06$$

Since $\frac{\sqrt{10}(\bar{X}-0.4)}{1} = Z \sim N(0,1)$, we know that

$$P\left(Z > \frac{\sqrt{10}(c - 0.4)}{1}\right) = 0.06$$

Since $P(Z > z_{0.94}) = 0.06$, where $z_{0.94} = 1.555$, we know that

$$\frac{\sqrt{10}(c - 0.4)}{1} = 1.555$$

and that

$$c = 0.4 + \frac{1.555}{\sqrt{10}} = 0.892$$

0.3.2 Result of the test

$\bar{x} = 0.54 < c = 0.892$, so we cannot reject at 6% the null hypothesis that the population mean is 0.4 against the alternative hypothesis that the population mean is higher than 0.4.

0.4 Two-sided Test:

Using a symmetric Two-sided Test of size 1%, test the null hypothesis that the population mean is 0.1 against the alternative hypothesis that the population mean is different than 0.1.

$$H_0: \quad \mu = 0.1$$
$$H_1: \quad \mu \neq 0.1$$

Decision Rule form: "Reject H_0 if $\bar{x} < c_1$ or if $\bar{x} > c_2$".

- We need to compute c_1 and c_2, which depend on the size of the test.

0.4.1 Computing c_1 and c_2

We need to find the value of c_1 and c_2 that satisfies the condition that the probability of type 1 error is 1%.

$$P\Big(\bar{X} < c_1 | \mu = 0.1\Big) + P\Big(\bar{X} > c_2 | \mu = 0.1\Big) = \alpha = 0.01$$

Since we are constructing a symmetric test, we have to satisfy the following two conditions:

$$P\Big(\bar{X} < c_1 | \mu = 0.1\Big) = 0.005 \qquad \text{and} \qquad P\Big(\bar{X} > c_2 | \mu = 0.1\Big) = 0.005$$

To compute c_1 and c_2 we need to find the distribution of the random variable $\bar{X}$. The random sample is normal, so we know that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. Under the null hypothesis $\bar{X} \sim N(0.1, \frac{1^2}{10})$, and therefore,

$$P\left(\frac{\sqrt{10}(\bar{X} - 0.1)}{1} < \frac{\sqrt{10}(c_1 - 0.1)}{1}\right) = 0.005 \quad \text{and} \quad P\left(\frac{\sqrt{10}(\bar{X} - 0.1)}{1} > \frac{\sqrt{10}(c_2 - 0.1)}{1}\right) = 0.005$$

Since $\frac{\sqrt{10}(\bar{X}-0.1)}{1} = Z \sim N(0,1)$, we know that

$$P\left(Z < \frac{\sqrt{10}(c_1 - 0.1)}{1}\right) = 0.005 \quad \text{and} \quad P\left(Z > \frac{\sqrt{10}(c_2 - 0.1)}{1}\right) = 0.005$$

Since $P(Z < z_{0.005}) = 0.005$ and $P(Z > z_{0.995}) = 0.005$, where $z_{0.005} = -2.575$ and $z_{0.995} = 2.575$, we know that

$$\frac{\sqrt{10}(c_1 - 0.1)}{1} = -2.575 \quad \text{and} \quad \frac{\sqrt{10}(c_2 - 0.1)}{1} = 2.575$$

and that

$$c_1 = 0.1 + \frac{-2.575}{\sqrt{10}} = -0.714 \quad \text{and} \quad c_2 = 0.1 + \frac{2.575}{\sqrt{10}} = 0.914$$

• Note that to compute c_2 we follow the same steps used to compute c in the One-sided Test done above (with $\alpha = 0.005$).

0.4.2 Result of the test

$\bar{x} = 0.54 > c_1 = -0.714$ and $\bar{x} = 0.54 < c_2 = 0.914$, so we cannot reject at 1% the null hypothesis that the population mean is 0.1 against the alternative hypothesis that the population mean is different than 0.1.

0.5 Generalized Likelihood Ratio Test (GLRT):

We will no solve this test, but we will just give the basic guidelines.

$$H_0: \quad \mu = 0$$
$$H_1: \quad \mu \neq 0$$

Decision Rule form: "Reject H_0 if $T > d$".

Where T is the following statistic:

$$T = \frac{\sup_{\theta \in \Omega_0} L(\theta_1, ..., \theta_k | x_1, ..., x_n)}{\sup_{\theta \in \Omega} L(\theta_1, ..., \theta_k | x_1, ..., x_n)} = \frac{\sup_{\theta \in \Omega_0} f(\mathbf{x} | \theta \in \Omega_0)}{\sup_{\theta \in \Omega} f(\mathbf{x} | \theta \in \Omega)}$$

• We need to compute T and d. For T, we need to find the maximum value of the likelihood function (given the realization of the sample) evaluating μ at all the possible values in the null and alternative set ($\forall \mu \in \Re$ in this case). This maximum value of the likelihood function is the number that goes in the denominator.

• We need to follow the same procedure to compute the value in the numerator, but we only need to evaluate the likelihood at all possible values of μ in the null space. Since the null space is singleton in this case, we have to evaluate the likelihood only at $\mu = 0$.

• We compute d using the following condition, $P(T < d | \mu = 0) = \alpha$. As with the case of the LR-Test, the strategy to determine the distribution of T depends on each case and many times it is be very hard to compute. We have an alternative when computing a GLRT, which applies only to the cases where we can assume that $n \to \infty$. In a large sample, we know that the limiting distribution of $-2lnT$ is (this result is not proved in this class):

$$-2lnT \overset{n \to \infty}{\sim} \chi^2_{(r)} \ ;$$

where r is the # of free parameters in Ω minus the # of free parameters in Ω_0. [1]

• Reject H_0 if $-2lnT > \chi^2_{(r),\alpha}$.

[1] The technical result says that the distribution is a $\chi^2_{(r)}$ with degrees of freedom $r = \dim\Omega - \dim\Omega_0$.