14.128. Problem Set #3

1 Neoclassical Growth: Linear and Non-Linear Speed of Convergence

Consider the neoclassical growth model with $u(c) = c^{1-\sigma}/(1-\sigma)$ $G(k,1) = k^{\alpha}$ and depreciation rate δ.

(a) Using the linearized dynamics compute several tables showing the speed of convergence as functions of the parameters α and σ (for $\beta = .97$ and $\delta = .1$). Use as a measure for the speed of convergence the half-life of the difference between capital and the steady state level of capital capital, i.e. $k_t - k_{ss}$. That is, find the time t for which $k_t - k_{ss} = \frac{1}{2}(k_0 - k_{ss})$, denote this value by $\tilde{t}$, in general $\tilde{t}$ will not be an integer. In the linearized model this number will not depend on k_0. [Hint: to do this quickly, create two vectors with the parameter values for σ and α that you want to use, then write a double loop into your code that goes over the different entries of these vectors; store the half-lifes into a matrix]

Now we will compute the actual non-linear dynamics and define a speed of convergence for it starting from some k_0 [with the non-linear dynamics our measure may depend on k_0].

Proceed as follows: find the actual non-linear policy function $k_{t+1} = g(k_t)$ by value function iteration numerically. Next compute a sequence for capital using g starting from k_0. Using this sequence find the smallest value of t such that $|k_t - k_{ss}| \leq \frac{1}{2}|k_0 - k_{ss}|$, denote this value by $\hat{t}$, and define:

$$\hat{\lambda} = \left(\frac{(k_{\hat{t}} - k_{ss})}{(k_0 - k_{ss})}\right)^{\frac{1}{\hat{t}}}$$

Next using this $\hat{\lambda}$ compute the half life of a system $x_{t+1} = \hat{\lambda} x_t$. This half-life is our summary statistic for the speed of convergence starting from k_0. In your calculations use $k_0 = \frac{1}{2}k_{ss}$ [note that k_{ss} and thus k_0 depends on the parameters].

(b) Perform this calculation for an interesting subset of the parameter values for which you computed the linear dynamics speed of convergence. Compare your results.

2 Two-Period Cycles

Do exercise 6.7 of SLP, page 157 (all parts: a, b, c, d, e and f).

3 Brock-Mirman

Consider the Brock-Mirman problem:

$$V^*(k_0, \theta_0) \equiv \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to $c_t + k_{t+1} \leq A k_t^\alpha \theta_t$, k_0 given, and where $\{\theta_t\}$ is an i.i.d. sequence with $\ln(\theta_t)$ distributed with density $h(\theta)$ with bounded support in $[\theta_L, \theta_H]$ ($A > 0$, $1 > \alpha > 0$).

The associated Bellman equation for this problem is:

$$V(k, \theta) = \max_{0 \leq k' \leq A k^\alpha \theta} \left\{ \ln(A k^\alpha \theta - k') + \beta \int_{\theta_L}^{\theta_H} V(k', \theta') h(\theta') dh \right\}.$$

(a) Verify that $V(k, \theta) = a_1 \log k + a_2 \log \theta + a_3$ solves the Bellman equation and compute a_1, a_2 and a_3 as functions of the parameters of the problem. Is this function the value function of the sequence problem, i.e. is $V^* = V$?

(b) What is the optimal policy rule for consumption? How does the optimal consumption rule change with β and α?

(c) Show that there is an alternative recursive formulation of this problem with a single state variable. In other words, produce a single state variable that is a sufficient statistic for the dynamic problem at any date t together with a functional equation that represents the problem using such a state variable. In such a formulation, can the state for $t + 1$ be chosen deterministically at t?

4 A Glimpse at Hyperbolic Discounting

An individual lives forever from $t = 0, 1, ..., \infty$. Think of the individual as actually consisting of different personalities, one for each period. Each personality is a distinct agent (time-t agent) with a distinct utility function and constraint set. Personality t has the following preferences

$$u(c_t) + \beta \sum_{j=1}^{\infty} \delta^j u(c_{t+j})$$

where $u(\cdot)$ is bounded twice differentiable, increasing and strictly concave function of consumption; $\beta \in (0, 1]$ and $\delta \in (0, 1)$. An individual with these preferences is called a **hyperbolic discounter**.

At each t, let there be a savings technology described by

$$k_{t+1} + c_t \leq f(k_t)$$

where f is a standard production function satisfying Inada conditions. There is no other source of income.

Assume that time$-t$ personality decides on consumption at time t only, and this consumption decision is function of k_t (i.e. $c(k_t)$) only. Assume that every time-t personality uses the same consumption function. Let

$$W(k_t) \equiv \beta \sum_{j=o}^{\infty} \delta^j u(c(k_{t+j}))$$

and where $\{k_{t+j}\}_{j=0}^{\infty}$ is defined recursively by $k_{t+j+1} = f(k_{t+j}) - c(k_{t+j})$, with k_t given.

A Markov equilibrium is then a function w that is a fixed point of the following functional operator T: to compute TW for any W we first find

$$c^*(k) \in \arg\max \{u(c) + \delta W(f(k) - c)\}$$

and then define

$$TW(k) \equiv \max_{c \in [0, f(k)]} \{u(c) + \delta W(f(k) - c)\} - (1 - \beta) u(c^*(k))$$

For any fixed point $w = Tw$ we may refer to the associated $c^*(k)$ as the equilibrium Markov strategy. (To avoid complications, assume that the set $\arg\max \{u(c) + \delta W(f(k) - c)\}$ is a singleton. We could modify things slightly to deal with the case where it isn't).

(a) What is the main conflict between different personalities? For which values of β do they all agree about the optimal plan?

(b) Interpret the operator T.

(c) If $\beta = 1$, is T a contraction mapping [*Hint: use Blackwell sufficient conditions for a contraction*]? How many (Markov) equilibria exist?

(d) For $\beta < 1$, can you say that T is a contraction mapping using Blackwell conditions? Can you say there is a unique (Markov) equilibrium? What is different between (c) and (d)?

(e) Suppose now that $u(c) = \log c$ and $f(k) = Ak^{\alpha}$ with $\alpha \in (0,1)$. Verify that one possible fixed point for T is of the form $W(k) = a \log k + b$. Determine a and b. What is the equilibrium consumption policy? How does it changes with β?

(f) (Observational Equivalence) Suppose there is another individual, call him an exponential consumer, with a $\beta^e = 1$ and $u^e(c) = \ln(c)$. Can you find a discount rate δ^e for this exponential consumer such that with $f(k) = Ak^{\alpha}$, the optimal consumption policy for this exponential consumer is the same as the equilibrium policy described in (e) for a hyperbolic consumer, with a given $\beta < 1$ and δ? What does this tell us about the ability to empirically separate a hyperbolic discounter from an exponential consumer?