Problem Set 1

1 Answers to the required problems

3.2

a) Take any three vectors x, y, z in R^l and two real number $\alpha, \beta \in R$. Define the zero vector $\theta = (0, .., 0) \in R^l$. To check that it is a vector space, define the sum of two vectors as the vector of the sum element by element; and the scalar multiplication as the multiplication of every element by an scalar. It is trivial to check that this operations are closed in a finite dimesional R^l. It is not very difficult then to check conditions a to h of the definition of a real vector space in page 43 of SLP by using the element by element operations. For example for property c we have that

$$\begin{aligned} \alpha(x+y) &= \alpha(x_1+y_1, ..., x_l+y_l) = (\alpha x_1 + \alpha y_1, ..., \alpha x_l + \alpha y_l) \\ &= \alpha x + \alpha y \end{aligned}$$

b) Straigthforward extension of part a)

c) Define the addition of two sequences as the element by element addition, and scalar multiplication as the multiplication of every element by the same real number. Then proceed as in part a.

d) Take $f, g : [a, b] \to R$ and $\alpha \in R$. Let $\theta(x) = 0$. Define the addition of functions by $(f+g)(x) = f(x) + g(x)$, and scalar multiplication by $(\alpha f)(x) = \alpha f(x)$. A function f is continuous if $x_n \to x$ implies that $f(x_n) \to f(x)$. To see that $f + g$ is continuous, take a sequence $x_n \to x$ in $[a, b]$. Then

$$\lim_{x_n \to x} (f+g)(x_n) = \lim_{x_n \to x} f(x_n) + g(x_n) = f(x) + g(x) = (f+g)(x)$$

Now you can proceed as in part c, checking that the properties are defined for every point of the function.

e) Take the vectors $(0,1)$ and $(1,0)$. Then $(1,0)+(0,1)$ is not in the unit circle.

f) Choose $\alpha \in (0,1)$. Then $1 \in I$ but $\alpha 1 \notin I$. Violates the definition of vector space

g) Let $f : [a,b] \to R_+$. Take $\alpha < 0$, then $\alpha f \leq 0$, so does not belong to the set of nonnegative functions.

3.3.

a) Take three different integers x, y,and z. The non-negative property holds trivially for the absolute value. Also

$$\rho(x,y) = |x-y| = |y-x| = \rho(y,x)$$

Finally

$$\rho(x,y) = |x-y| \leq |x-z| + |z-y| \leq \rho(x,z) + \rho(z,y)$$

b) First, $\rho(x,y) \geq 0$, and with equality only when $x = y$. It is also true that $\rho(x,y) = \rho(y,x)$.

Finally to show that $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$, notice that you have to consider three cases, when $z = y$, $z = x$ and when $z \notin \{x,y\}$. For the first two cases the triangle inequality holds with equality. For the last one it holds with inequality $\rho(x,y) < 2$ for all x, y.

c) Take three functions $x(t), y(t)$, and $z(t)$. The first two properties of the metric are immediate from the definition of absolute value. Notice also that x, y continuous in $[a,b]$ implies that the functions are bounded. The proposed metric is then real valued (not extended real). To prove the triangle inequality let

$$\begin{aligned}
\rho(x,z) &= \max_{t\in[a,b]} |x(t) - z(t)| = \max_{t\in[a,b]} |x(t) - y(t) + y(t) - z(t)| \\
&\leq \max_{t\in[a,b]} \{|x(t) - y(t)| + |y(t) - z(t)|\} \\
&\leq \max_{t\in[a,b]} |x(t) - y(t)| + \max_{t\in[a,b]} |y(t) - z(t)| = \rho(x,y) + \rho(y,z)
\end{aligned}$$

d) Very similar to c.

e) Similar to a.

f) The first two properties come from the absolute value plus $f(0) = 0$, and f is strictly increasing (there is only one zero).

To see the triangle inequality, let

$$\rho(x,y) = f(|x-y|) = f(|x-z+z-y|)$$

By strictlt increasing f we have

$$f(|x-z+z-y|) \leq f(|x-z|+|z-y|)$$

And by concavity we have

$$f(|x-z|+|z-y|) \leq f(|x-z|) + f(|z-y|) = \rho(x,z) + \rho(z,y)$$

3.4.

a) The norm is non-negative comes from the definition of the square root of sum of squares. The second one, $||\alpha x|| = |\alpha|\ ||x||$, is implied by

$$||\alpha x||^2 = \sum (\alpha x_i)^2 = \alpha^2 \sum x_i^2 = \alpha^2\, ||x||$$

To prove the triangle inequality we follow :

$$||x+y||^2 = \sum (x_i+y_i)^2 = \sum x_i^2 + 2\sum x_i y_i + \sum y_i^2$$

By Cauchy-Schwartz we have that $\sum x_i y_i \leq [(\sum x_i^2)(\sum y_i^2)]^{1/2}$ so

$$\begin{aligned} \sum x_i^2 + 2\sum x_i y_i + \sum y_i^2 &\leq \sum x_i^2 + 2\left[\left(\sum x_i^2\right)\left(\sum y_i^2\right)\right]^{1/2} + \sum y_i^2 \\ &= ||x||^2 + 2\,||x||\ ||y|| + ||y||^2 = (||x|| + ||y||)^2 \end{aligned}$$

This implies that $||x+y|| \leq ||x|| + ||y||$.

b),c) are very similar exploiting the absolute value properties.

d) The proposed norm is non-negative and is real valued (bounded sequences). The first property, notice that $||x|| = 0$ only if for all k, $x_k = 0$. The second property we have $||\alpha x|| = \sup_k |\alpha x_k| = |\alpha| \sup_k |x_k| = |a|\ ||x||$. For the triangle inequality we have

$$\begin{aligned} ||x+y|| &= \sup_k (|x_k + y_k|) \leq \sup_k (|x_k| + |y_k|) \leq \sup_k |x_k| + \sup_k |y_k| \\ &= ||x|| + ||y|| \end{aligned}$$

e) Proceed as in part d.

f) Proceed as in part d.

3.5.

a) By the triangle inequality we know that

$$\rho(x, y) \leq \rho(x_n, x) + \rho(x_n, y)$$

given that $x_n \to x$, and $x_n \to y$, the right hand side can be made as small as wanted, implying that $\rho(x, y) = 0$ and $x = y$.

b) If $x_n \to x$, then for any ε, we can find N such that $\rho(x_{n_1}, x) < \varepsilon/2$ for $n_1 > N$. The distance between to x_{n_1} and x_{n_2} with $n_2 > N$ is then $\rho(x_{n_1}, x_{n_2}) \leq \rho(x_{n_1}, x) + \rho(x_{n_2}, x) = \varepsilon$ for all $n_1, n_2 > N$. (which is the Cauchy definition).

c) Pick an ε, then there exists an N such that for $n_1 > N$ we have that $\rho(x_{n_1}, x_{n_2}) < \varepsilon$. Then $\rho(x_{n_1}, 0) \leq \rho(x_{n_1}, x_N) + \rho(x_N, 0) = \varepsilon + \rho(x_N, 0)$. Let $M = \max_{n<N} \{\rho(x_n, 0)\}$. So we have then that for all n,

$$\rho(x_n, 0) \leq \max\{\varepsilon + \rho(x_N, 0), M\} < \infty \text{ for all } n$$

so the sequence is bounded.

d) The fact that $x_n \to x$ implies that every subsequence converges is easy. To show that if every subsequence converges to x, then $\{x_n\}$ converges to x we need to show that if x_n does not converge to x, then there is a subsequence that does not converge to x. We can construct such a sequence by showing that if $\{x_n\}$ does not converge, for any ε, and for any N_1 we can find an $k_1 > N$ such that $|x_{k_1} - x| > \varepsilon$. Let $N_2 > k_1$ and can find a k_2 such that $|x_{k_2} - x| > \varepsilon$ and so on , and construct a sequence of $\{x_{k_i}\}$ that is always bounded away from x by ε.

3.6.

a) The metric in 3.3a is complete. Just choose $\varepsilon < 1$, such that $|x_n - x_m| < \varepsilon = 1$. This implies that $x_n = x_m = x$. Where x is the limit of the sequence. The 3.3b metric is similar.

The metric in 3.4a is complete. To show that, notice that if $\{x_n\}$ is a Cauchy sequence under the norm, then each of the k_{th} elements of x_n are also a Cauchy sequence. Given that the real line is complete, each sequence of k_{th} elements converges to some x^k. Define $x = \left(x^1, ..., x^k\right)$. Compute the distance from x_n to x. This is $\rho(x_n, x)^2 = \sum \left(x_n^i - x^i\right)^2$. Given that every

$x_n^i \to x^i$, then $\rho(x_n, x)^2 \to 0$ which proves the limit result. The metric in 3.4b and 3.4c is similar.

For 3.4d. Let x_n^k be the kth element of the n sequence. The fact that a sequence of sequences is Cauchy (under the norm) implies that $\{x_n^k\}$ is Cauchy as well. This implies that $\{x_n^k\}$ converges to some x^k. Let $x = \{x^1, x^2, ...\}$. It is easy to see that $\rho(x_n, x) \leq \rho(x_n, x_m) + \rho(x_m, x) \leq \rho(x_n, x_m) + \sup_k \{\rho(x_m^k, x^k)\}$. Given that $x_m^k \to x^k$ and $\{x_n\}$ is Cauchy, we can make the left hand side of the inequality as small as possible, implying that $\rho(x_n, x) \to x$.

3.4.e. The proof was done in class for more general space.

The 3.3c is not complete. Just analyze the limit of for example $x_n(t) = 1 + a_n t$ as $a_n \to 0$ with $a_n > 0$. This limit is just $x_n(t) = 1$, which is not strictly increasing.

3.3e is not complete: rational sequences can converge to irrational numbers.

In 3.4.f, just think of the counter example for $a = 0$,$b = 1$ and $x_n(t) = t^n$. In this case, $\rho(x_n, x_m) \to 0$, but the limit of $x_n(t)$ is discountinuos at $t = 0$.

For the case of 3.3.c replacing with "non-decreasing" we can construct the limit function point by point as before. Now, we have to show that the limit function is non-decreasing, to do that suppose that is not, that the Cauchy sequence of $\{f_n\}$ converges to f , but for some $t' > t$ we have that $f(t) - f(t') > \varepsilon$. Then

$$\begin{aligned} 0 &< \varepsilon < f(t) - f(t') = f(t) - f_n(t) + [f_n(t) - f_n(t')] + f_n(t') - f(t') \\ &\leq 2\,||f - f_n|| + f_n(t) - f_n(t') \end{aligned}$$

Given that $f \to f_n$, it has to be the case that $f_n(t) - f_n(t') > \varepsilon$, a contradiction of non-decreasing property of f_n.

b) Since S' is closed in S, any convergent sequence in S' converges in S'. Given that any Cauchy sequence in S converges in S (by completeness), implies that any Cauchy sequence in S' converges, and hence converges in S'.

3.9. See that

$$\begin{aligned} \rho(T^n v_0, v) &\leq \rho\left(T^n v_0, T^{n+1} v_0\right) + \rho\left(T^{n+1} v_0, v\right) = \\ &= \rho\left(T^n v_0, T^{n+1} v_0\right) + \rho\left(T^{n+1} v_0, Tv\right) \\ &\leq \rho\left(T^n v_0, T^{n+1} v_0\right) + \beta\rho(T^n v_0, Tv) \end{aligned}$$

Rearranging term we get the inequility we need.

3.13.

a) Same as part b. with $f(x) = x$.

b) Choose any x. Since $0 \in \Gamma(x)$, $\Gamma(x)$ is non-empty. Choose any $y \in \Gamma(x)$ and consider the sequence $x_n \to x$. Let $\gamma \equiv y/f(x) \leq 1$ and $y_n = \gamma f(x_n)$. So $y_n \in \Gamma(x_n)$. Then by continuity of f we have that $\lim y_n = \gamma \lim f(x_n) = \gamma f(x) = y$. Hence f is l.h.p.c. at x.

Given x, $\Gamma(x)$ is compact valued. Take arbitrary sequences $x_n \to x$ and $y_n \in \Gamma(x_n)$. Given that $\{x_n\}$ converges, this implies that $\{x_n\}$ is bounded. This implies that $\{y_n\}$ is bounded as well. Because any bounded sequence of real numbers has a convergent subsequence there exists a convergent subsequence $\{y_{n_k}\}$. Let $y = \lim y_{n_k}$. Now we need to show that $\lim y_{n_k} \leq f(x)$. Suppose no, then for some N, we have that $y_{n_k} - f(x) > \varepsilon$ for all $n_k > N$. This implies then that $y_{n_k} - f(x_{n_k}) > 2\varepsilon$ for all $n_k > M > N$. But this is impossible, because $y_{n_k} \in \Gamma(x_{n_k})$.

c) Proceed coordinate by coordinate as in b)

4.3)

a) Let $v(x_0)$ be finite. Since v satisfies the FE, as shown in the proof of Theorem 4.3, for every $x_0 \in X$ and every $\varepsilon > 0$ there exists an $\underrightarrow{x} \in \Pi(x_0)$ such that

$$v(x_0) \leq u_n\left(\underrightarrow{x}\right) + \beta^{n+1} v(x_{n+1}) + \frac{\varepsilon}{2}$$

Taking the limit as $n \to \infty$ this gives

$$v(x_0) \leq u\left(\underrightarrow{x}\right) + \limsup_{n\to\infty} \beta^{n+1} v(x_{n+1}) + \frac{\varepsilon}{2} \leq u\left(\underrightarrow{x}\right) + \frac{\varepsilon}{2}$$

Given that $u\left(\underrightarrow{x}\right) \leq v^*(x_0)$ this gives us that

$$v(x_0) \leq v^*(x_0) + \varepsilon/2$$

and hence $v(x_0) \leq v^*(x_0)$ for all x_0.

If $v(x_0) = -\infty$, the result is immediate. If $v(x_0) = \infty$ the proof is along the lines of the last part of theorem 4.3

b) By the argument in theorem 4.3 we have that

$$\begin{aligned} v(x_0) &\geq u_n\left(\underset{\rightarrow}{x}\right) + \beta^{n+1} v(x_{n+1}) \\ v(x_0) &\geq \lim_{n\to\infty} u_n\left(\underset{\rightarrow}{x'}\right) + \lim_{n\to\infty} \beta^n v\left(x'_{n+1}\right) = u_n\left(\underset{\rightarrow}{x'}\right) \geq u\left(\underset{\rightarrow}{x}\right) \end{aligned}$$

For all $\underset{\rightarrow}{x} \in \Pi(x_0)$. This implies that

$$v(x_0) \geq v^*(x_0) = \sup_{\underset{\rightarrow}{x} \in \Pi(x_0)} u\left(\underset{\rightarrow}{x}\right)$$

and together with the result in part a), we have that $v = v^*$.

4.4)

a) Let K be a bound on F and M be a bound on f.Then

$$(Tf)(x) \leq K + \beta M, \text{ for all } x \in X$$

Hence $T : B(X) \to B(X)$.

We show now that T is a contraction mapping.

Monotonicity :

Let f, g with $f \leq g$. Then

$$\begin{aligned} (Tf)(x) &= \max_{y \in \Gamma(x)} F(x,y) + \beta f(y) = F(x, y^*) + \beta f(y^*) \\ &\leq F(x, y^*) + \beta g(y^*) \leq (Tg)(x) \end{aligned}$$

where $y^* \in \arg\max F(x,y) + \beta f(y)$

Discounting: It is easy to show that $T(f+a)(x) = (Tf)(x) + \beta a$.

So, T is a contraction. There is a unique fixed point. $\Gamma(x)$ is non-empty and finite-valued for all x implies that the optimal policy correspondence is non-empty; and the maximum is always attained.

b) Similar to part (a)

c) Note that

$$\begin{aligned} w_n(x) &= (T_{h_n} w_n)(x) \\ &\leq \max\left[F(x,y) + \beta w_n(y)\right] \\ &= (Tw_n)(x) = \left(T_{h_{n+1}} w_n\right)(x) \end{aligned}$$

So we have that $w_n \leq Tw_n$. Monotonicity of T_{h_n} implies that $T_{h_n} w_n \leq T_{h_n}(Tw_n) = T_{h_n}^2 w_n$. Iterating in this operator we have that

$$Tw_n \leq T_{h_{n+1}}^N w_n$$

But $w_{n+1} = \lim_{N\to\infty} T^N_{h_{n+1}} w_n$. Hence $Tw_n \leq w_{n+1}$. And

$$w_0 \leq Tw_0 \leq w_1 \leq Tw_1 \leq ... \leq Tw_n \leq v$$

By the contraction mapping,

$$\begin{aligned} ||w_n - v|| &\leq ||Tw_{n-1} - v|| \leq \beta\,||w_{n-1} - v|| \\ &\leq \beta\,||Tw_{n-2} - v|| \leq \beta^2\,||w_{n-2} - v|| \\ &\leq \beta^n\,||w_0 - v|| \end{aligned}$$

and hence $w_n \to v$ as $n \to \infty$.