Expected Utility

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In class some of you asked about the relation between the continuity axiom and the topology on the set of lotteries. The purpose of *Herstein and Milnor (1953): "An Axiomatic Approach to Measurable Utility"* was precisely to have a version of the Von Neumann-Morgenstern Theorem without imposing topological assumptions on the set of lotteries. Their idea was to use the convex structure of the set of alternatives to state the continuity axiom.

Let X be a convex set of alternatives. Let $\succsim$ be a binary relation on X, that is, a subset of $X \times X$, reporting the preferences of the DM (Decision Maker). We say that a function $U : X \to \mathbb{R}$ *represents* $\succsim$ if for all $x, y \in X$

$$x \succsim y \quad \Leftrightarrow \quad U(x) \geqslant U(y).$$

Moreover, we say that U is *linear*[1] if for all $x, y \in X$ and $\alpha \in [0, 1]$

$$U(\alpha x + (1 - \alpha)y) = \alpha U(x) + (1 - \alpha)U(y).$$

Herstein and Milnor consider the following set of axioms (I will refer to axioms as A1,...):

Axiom 1 (Weak Order)**.** $\succsim$ *is complete and transitive*

Remark 1 (Transitivity). No one will ever consider a preference relation which is not transitive. However: Let $X = [0, \infty)$ be dollars. Assume that $\succsim$ on X is transitive. Furthermore, assume that there is $\epsilon > 0$ such that, whenever $|x - y| \leqslant \epsilon$, we have that $x \sim y$. You can check that, no matter how small ϵ is, the DM is indifferent between zero and one billion dollars.

Remark 2 (Completeness). Incomplete preferences are much more interesting: if two alternatives are very different from each other, the DM may not be able to rank them. *Aumann (1962): "Utility Theory without the Completeness Axiom"* was the first to develop a theory of incomplete preferences. *Dubra, Maccheroni and Ok (2004): "Expected "Utility Theory without the Completeness Axiom"* characterizes incomplete preferences represented by a set of utilities $\mathcal{U} \subset \mathbb{R}^X$:

$$x \succsim y \quad \Leftrightarrow \quad U(x) \geqslant U(y) \quad \text{for all } U \in \mathcal{U}.$$

[1] It would be more proper to say that U is *affine*.

Axiom 2 (Continuity)**.** *For all $x, y, z \in X$, the sets $\{\alpha \in [0,1] : \alpha x + (1-\alpha)y \succsim z\}$ and $\{\beta \in [0,1] : z \succsim \beta x + (1-\beta)y\}$ are closed.*

This version of the continuity axiom uses the topology of the unit interval and the convex structure of X, instead of putting a topology on X. You can check that the continuity axiom we saw in class implies A2.[2]

Axiom 3 (Independence)**.** *For all $x, y, z \in X$, if $x \sim y$, then $\frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}x + \frac{1}{2}x$.*

This version of the independence axiom is much weaker than the version we saw in class. Proving the representation theorem under A3 is indeed much harder:

Theorem 1 (Herstein and Milnor (1953))**.** *The following statements are equivalent:*

(i) $\succsim$ on X satisfies A1, A2 and A3.

(ii) There exists a linear $U : X \to \mathbb{R}$ which represents $\succsim$.

This is known as the *Mixture Space Theorem.*[3] We obtain the expected utility criterion as a corollary. Let C be a finite set of consequences, and consider $\Delta(C)$, the set of lotteries. It is clear that $\Delta(C)$ is a convex set.

Corollary 1 (Expected Utility)**.** *The following statements are equivalent:*

(i) $\succsim$ on $\Delta(C)$ satisfies A1, A2 and A3.

(ii) There exists $u : C \to \mathbb{R}$ such that for all $p, q \in \Delta(C)$

$$p \succsim q \quad \Leftrightarrow \quad \sum_{c \in C} p(c)u(c) \geqslant \sum_{c \in C} q(c)u(c).$$

Proof. To see that (ii) implies (i), define $U : \Delta(C) \to \mathbb{R}$ such that for all $p \in \Delta(C)$

$$U(p) = \sum_{c \in C} p(c)u(c).$$

It is clear that U is linear and represents $\succsim$. By the Mixture Space Theorem, $\succsim$ satisfies A1, A2 and A3, which means that (i) holds.

On the other hand, to show that (i) implies (ii), observe that by the Mixture Space Theorem there is $U : \Delta(C) \to \mathbb{R}$ which is linear and represents $\succsim$. For every $c \in C$, denote by $\delta_c \in \Delta(C)$ the lottery putting probability one on c. Notice that any $p \in \Delta(C)$ can be written as $\sum_{c \in C} p(c)\delta_c$. This means that since U is linear

$$U(p) = \sum_{c \in C} p(c)U(\delta_c).$$

[2]Indeed, any meaningful topology on X has the property that: if $\alpha_n \to \alpha$ in $[0,1]$, then $\alpha_n x + (1-\alpha_n)y \to \alpha x + (1-\alpha)y$ in X.

[3]Herstein and Milnor consider for the set of alternatives an abstract notion of convex set, which they call *mixture set*, hence the name of the theorem.

Define $u : C \to \mathbb{R}$ such that $u(c) = U(\delta_c)$ for all $c \in C$. Since U represents $\succsim$, for all $p, q \in \Delta(C)$

$$p \succsim q \quad \Leftrightarrow \quad \sum_{c \in C} p(c)u(c) = U(p) \geqslant U(q) = \sum_{c \in C} q(c)u(c),$$

which means that (ii) holds, concluding the proof. $\square$

We are gonna prove a much simpler version of the Mixture Space Theorem under the stronger version of the independence axiom we saw in class:

Axiom 4. *For all $x, y, z \in X$ and $\alpha \in (0, 1)$*

$$x \succsim y \quad \Leftrightarrow \quad \alpha x + (1 - \alpha)z \succsim \alpha y + (1 - \alpha)z.$$

You can verify the following immediate implications of A4:

Lemma 1. *Assume that $\succsim$ on X satisfies A4. Then for all $x, y, z \in X$ and $\alpha \in (0, 1)$*

$$x \succ y \quad \Leftrightarrow \quad \alpha x + (1 - \alpha)z \succ \alpha y + (1 - \alpha)z,$$
$$x \sim y \quad \Leftrightarrow \quad \alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z.$$

I will first prove one direction of the Mixture space Theorem under the assumption that there are "extreme" alternatives:

Axiom 5. *There are $x_*, x^* \in X$ such that $x^* \succsim x \succsim x_*$ for all $x \in X$.*

Axiom A5 makes life easier because it guarantees that X is a preference interval. For $x, y \in X$, the *preference interval* is the set

$$[x, y] := \{z \in X : x \succsim z \succsim y\}.$$

If $\succsim$ satisfies A5, then $X = [x_*, x^*]$. Notice that, if $\succsim$ satisfies A4, then preference intervals are convex subset of X (why?).

Proposition 1. *Assume that $\succsim$ on X satisfies A1, A2, A4 and A5. Then there exists a linear function $U : X \to \mathbb{R}$ which represents $\succsim$.*

Once we have Proposition 1, it is not hard to drop A5, and prove the Mixture Space Theorem with A4 instead of A3:

Proposition 2. *The following statements are equivalent:*

(i) $\succsim$ *on X satisfies A1, A2 and A4.*

(ii) There exists a linear $U : X \to \mathbb{R}$ which represents $\succsim$.

In the next two sections we are gonna see the proofs of these two propositions. The proof of Proposition 1 is what you need to remember (it is just a rephrasing of the proof for expected utility that you have in the lecture notes). Proposition 2, instead, it is only for your curiosity. In fact, as long as we are concern with $X = \Delta(C)$, lotteries with a finite set of consequences, it can be shown that A5 is implied by A1, A2 and A4:

Exercise 1. Assume that $\succsim$ on $\Delta(C)$ satisfies A1, A2 and A4. Then $\succsim$ satisfies also A5.

Proof of Proposition 1

Without loss of generality, assume that $x^* \succ x_*$ (if not, set $U(x) = 0$ for all $x \in X$). The idea of the proof is to measure the utility of alternative x relative to the utility of the "extreme" alternatives x_* and x^* using the convexity of X. If $x = \alpha x_* + (1-\alpha)x^*$ for some $\alpha \in [0,1]$, it is natural to assign utility α to x, and this would be a good idea to get linearity (why?). This idea, however, makes sense only if bigger weights on x^* are preferred by the DM. Thanks to independence axiom, this is the case:

Lemma 2. *Take $\alpha, \beta \in [0,1]$ such that $\alpha > \beta$. Then $\alpha x^* + (1-\alpha)x_* \succ \beta x^* + (1-\beta)x_*$.*

Proof. If $\alpha = 1$ and $\beta = 0$, there is nothing to do. Assume now that $\alpha = 1$ and $\beta \in (0,1)$. Since $x^* \succ x_*$

$$\alpha x^* + (1-\alpha)x_* = x^* = \beta x^* + (1-\beta)x^* \succ \beta x^* + (1-\beta)x_*,$$

where the last "inequality" holds by A4. Assume now that $\beta = 0$ and $\alpha \in (0,1)$. Since $x^* \succ x_*$

$$\alpha x^* + (1-\alpha)x_* \succ \alpha x_* + (1-\alpha)x_* = x_* = \beta x^* + (1-\beta)x_*,$$

where the first inequality holds by A4. We are left with the case $\alpha, \beta \in (0,1)$. Notice that $\beta/\alpha \in (0,1)$, and therefore we can write

$$\alpha x^* + (1-\alpha)x_* = \frac{\beta}{\alpha}(\alpha x^* + (1-\alpha)x_*) + (1 - \frac{\beta}{\alpha})(\alpha x^* + (1-\alpha)x_*).$$

We have shown above that $\alpha x^* + (1-\alpha)x_* \succ x_*$. So we can use A4 once again to get that

$$\frac{\beta}{\alpha}(\alpha x^* + (1-\alpha)x_*) + (1 - \frac{\beta}{\alpha})(\alpha x^* + (1-\alpha)x_*) \succ \frac{\beta}{\alpha}(\alpha x^* + (1-\alpha)x_*) + (1 - \frac{\beta}{\alpha})x_*.$$

Luckily

$$\frac{\beta}{\alpha}(\alpha x^* + (1-\alpha)x_*) + (1 - \frac{\beta}{\alpha})x_* = \beta x^* + (1-\beta)x_*,$$

which concludes the proof. □

We cannot hope to write any alternative x as convex combinations of x_* and x^*. However, since we have assumed that $x^* \succsim x \succsim x_*$ (read A5), we can use the continuity axiom to show that x is indifferent to a convex combination of x_* and x^*.

Lemma 3. *For any $x \in X$, there is $\alpha_x \in [0,1]$ such that $x \sim \alpha_x x_* + (1-\alpha_x)x^*$.*

Proof. Consider the set

$$A = \{\alpha \in [0,1] : \alpha x^* + (1-\alpha)x_* \succsim x\} \quad \text{and} \quad B = \{\beta \in [0,1] : x \succsim \beta x^* + (1-\beta)x_*\}.$$

Since $\succsim$ is complete, $A \cup B = [0,1]$. Furthermore by A5 $x^* \in A$ and $x_* \in B$, and therefore both A and B are nonempty. Finally, by A2, A and B are closed. Since the interval $[0,1]$ is connected, a basic result from real analysis tells us that $A \cap B \neq \varnothing$. Therefore we can pick $\alpha_x \in A \cap B$ and be sure that $x \sim \alpha_x x_* + (1-\alpha_x)x^*$, as wanted. □

Thanks to Lemma 1 and 2, we can well define the function $U : X \to \mathbb{R}$ such that, for all x, $U(x) = \alpha_x$ and be sure that it represents $\succsim$.[4] The only remaining thing to prove is that U is linear. This is another consequence of the independence axiom:

Lemma 4. *The function U is linear.*

Proof. Take $x, y \in X$ and $\lambda \in (0,1)$. We wish to show that

$$\alpha_{\lambda x+(1-\lambda)y} = U(\lambda x + (1-\lambda)y) = \lambda U(x) + (1-\lambda)U(y) = \lambda\alpha_x + (1-\lambda)\alpha_y.$$

By Lemma 2, we know there can be at most one $\alpha \in [0,1]$ such that $\lambda x+(1-\lambda)y \sim \alpha x^*+(1-\alpha)x_*$. Hence, to show that $\alpha_{\lambda x+(1-\lambda)y} = \lambda\alpha_x + (1-\lambda)\alpha_y$, it is enough to show that

$$\begin{aligned} \lambda x + (1-\lambda)y &\sim (\lambda\alpha_x + (1-\lambda)\alpha_y)x^* + (1-(\lambda\alpha_x + (1-\lambda)\alpha_y))x_* \\ &= \lambda(\alpha_x x^* + (1-\alpha_x)x_*) + (1-\lambda)(\alpha_y x^* + (1-\alpha_y)x_*). \end{aligned}$$

Since $x \sim \alpha_x x^* + (1-\alpha_x)x_*$ and $y \sim \alpha_y x^* + (1-\alpha_y)x_*$, by A4 and transitivity

$$\begin{aligned} \lambda x + (1-\lambda)y &\sim \lambda(\alpha_x x^* + (1-\alpha_x)x_*) + (1-\lambda)y \\ &\sim \lambda(\alpha_x x^* + (1-\alpha_x)x_*) + (1-\lambda)(\alpha_y x^* + (1-\alpha_y)x_*), \end{aligned}$$

as wanted. □

Proof of Proposition 2

It is easy to see that (ii) implies (i). Therefore let's focus on the other direction: (ii) implies (i). Without loss of generality, assume that there are $x^*, x_* \in X$ such that $x^* \succ x_*$ (if not, just set $U(x) = 0$ for all $x \in X$). Take any pair $x, y \in X$ such that $x \succsim x^* \succ x_* \succsim y$. Applying Proposition 1 to $\succsim$ on $[y, x]$, we can find a linear function $U_{x,y} : [y, x] \to \mathbb{R}$ which represents $\succsim$

[4] To check that U is well defined, we need to verify that there exists a *unique* α_x such that $x \sim \alpha_x x_* + (1-\alpha_x)x^*$. Existence comes from Lemma 3, while uniqueness from Lemma 2.

on $[y,x]$.[5] Furthermore, we can choose U_{xy} such that $U_{x,y}(x_*) = 0$ and $U_{x,y}(x^*) = 0$ (why?). Now define $U : X \to \mathbb{R}$ such that

$$U(z) = U_{x,y}(z) \quad \text{for } x, y \in X \text{ such that } x_*, x^*, z \in [y,x].$$

We need to prove that (i) U is well defined, (ii) U represents $\gtrsim$ on X, and (iii) U is linear.

Lemma 5. *U is well defined.*

Proof. Fix $z \in X$. Take $x, y, x', y' \in X$ such that $x_*, x^*, z \in [y,x], [y',x']$. We wish to show that $U_{x,y}(z) = U_{x',y'}(z)$: if so, U is well defined. There are three cases to consider: $z \gtrsim x^*$, $z \in [x_*, x^*]$ and $x_* \gtrsim z$. We will only consider the case $z \gtrsim x^*$: the other two cases can be treated analogously. Assume therefore that $x^* \in [z, x_*]$. Since $U_{x,y}$ represents $\gtrsim$ on $[z, x_*]$,

$$U_{x,y}(x^*) \in (U_{x,y}(x_*), U_{x,y}(z)].$$

This means that we can find $\alpha \in (0,1]$ such that

$$U_{x,y}(x^*) = \alpha U_{x,y}(z) + (1-\alpha) U_{x,y}(x_*).$$

Since we have chosen $U_{x,y}(x^*) = 1$ and $U_{x,y}(x_*) = 0$, we obtain $U_{x,y}(z) = 1/\alpha$. Furthermore, since $U_{x,y}$ is linear on $[z, x_*]$:

$$U_{x,y}(x^*) = \alpha U_{x,y}(z) + (1-\alpha) U_{x,y}(x_*) = U_{x,y}(\alpha z + (1-\alpha) x_*).$$

Hence, since $U_{x,y}$ represents $\gtrsim$ on $[z, x_*]$, we get that $x^* \sim \alpha z + (1-\alpha) x_*$. But it is also true that $U_{x',y'}$ represents $\gtrsim$ on $[z, x_*]$, and therefore

$$U_{x',y'}(x^*) = U_{x',y'}(\alpha z + (1-\alpha) x_*).$$

Since $U_{x',y'}$ is linear on $[z, x_*]$ and we have chosen $U_{x',y'}(x^*) = 1$ and $U_{x',y'}(x_*) = 0$, we easily get that also $U_{x',y'}(z) = 1/\alpha$, and therefore $U_{x,y}(z) = U_{x',y'}(z)$, as wanted. □

Lemma 6. *U represents $\gtrsim$.*

Proof. Take $z, z' \in X$. We can certainly find a preference interval $[y,x]$ big enough so that $x^*, x_*, z, z' \in [y,x]$. Then, using the fact that $U_{x,y}$ represents $\gtrsim$ on $[y,x]$:

$$z \gtrsim z' \quad \Leftrightarrow \quad U(z) = U_{x,y}(z) \geqslant U_{x,y}(z') = U(z'),$$

as wanted. □

Lemma 7. *U is linear.*

[5] Note that $\gtrsim$ on $[y,x]$ satisfies also A5.

Proof. Take $z, z' \in X$ and $\alpha \in [0,1]$. We can certainly find a preference interval $[y, x]$ big enough so that $x^*, x_*, z, z' \in [y, x]$. Since $[y, x]$ is convex, $\alpha z + (1-\alpha)z' \in [y, x]$. Then, using the fact that $U_{x,y}$ is linear:

$$U(\alpha z + (1-\alpha)z') = U_{x,y}(\alpha z + (1-\alpha)z') = \alpha U_{x,y}(z) + (1-\alpha)U_{x,y}(z') = \alpha U(z) + (1-\alpha)U(z'),$$

as wanted. $\square$