Attitudes Towards Risk

14.123 Microeconomic Theory III
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Model

- C = R = wealth level
- Lottery = cdf F (pdf f)
- Utility function u : R→R, increasing
- $U(F) \equiv E_F(u) \equiv \int u(x)dF(x)$
- $E_F(x) \equiv \int x dF(x)$

Attitudes Towards Risk

DM is

- risk averse if $E_F(u) \leq u(E_F(x))$ $(\forall F)$
- strictly risk averse if $E_F(u) < u(E_F(x))$ ($\forall$ "risky" F)
- risk neutral if $E_F(u) = u(E_F(x))$ $(\forall F)$
- risk seeking if $E_F(u) \geq u(E_F(x))$ $(\forall F)$

DM is

- risk averse if u is concave
- strictly risk averse if u is strictly concave
- risk neutral if u is linear
- risk seeking if u is convex

Certainty Equivalence

- $CE(F) = u^{-1}(U(F))=u^{-1}(E_F(u))$
- DM is
  - risk averse if $CE(F) \leq E_F(x)$ for all F;
  - risk neutral if $CE(F) = E_F(x)$ for all F;
  - risk seeking if $CE(F) \geq E_F(x)$ for all F.
- Take DM1 and DM2 with u_1 and u_2.
- DM1 is more risk averse than DM2
  - $\Leftrightarrow$ u_1 is more concave than u_2, i.e.,
  - $\Leftrightarrow$ $u_1 = g \circ u_2$ for some concave function g,
  - $\Leftrightarrow$ $CE_1(F) \equiv u_1^{-1}(E_F(u_1)) \leq u_2^{-1}(E_F(u_2)) \equiv CE_2(F)$

Absolute Risk Aversion

- absolute risk aversion:

  $r_A(x) = -u''(x)/u'(x)$
- constant absolute risk aversion (CARA)

  $u(x) = -e^{-\alpha x}$
- If $x \sim N(\mu,\sigma^2)$, $CE(F) = \mu - \alpha\sigma^2/2$
- Fact: More risk aversion ⇔ higher absolute risk aversion everywhere
- Fact: Decreasing absolute risk aversion (DARA)

  ⇔ $\forall y>0$, u_2 with $u_2(x) \equiv u(x+y)$ is less risk averse

Relative risk aversion:

- relative risk aversion:

  $r_R(x) = -xu''(x)/u'(x)$
- constant relative risk aversion (CRRA)

  $u(x) = x^{1-\rho}/(1-\rho)$,
- When $\rho = 1$, $u(x) = \log(x)$.
- Fact: Decreasing relative risk aversion (DRRA)

  ⇔ $\forall t>1$, u_2 with $u_2(x) \equiv u(tx)$ is less risk averse

Optimal Risk Sharing

- $N = \{1,\ldots,n\}$ set of agents
- S = set of states s
- Each i has a concave utility function u_i & an asset that pays $\underline{x}_i(s)$
- A = set of allocations $x = (x_1,\ldots, x_n)$ s.t. for all s,

  $$x_1(s)+\ldots+ x_n(s) \leq \underline{x}_1(s)+\ldots+ \underline{x}_n(s) \equiv X(s) \quad (*)$$
- $V = E[u(A)]$ and $\underline{V}$ = comprehensive closure of V, convex
- x^* = a Pareto-optimal allocation, $v^* = u(x^*)$
- Since $\underline{V}$ is convex, $v^* \in \text{argmax}_{v\in V}\ \lambda_1 v_1+\ldots+ \lambda_n v_n$ for some $(\lambda_1,\ldots,\lambda_n)$
- i.e. $x^* \in \text{argmax}_{x\in A}\ E[\lambda_1 u_1(x_1) +\ldots+ \lambda_n u_n(x_n)]$
- For every s, $x^*(s)$ maximizes $\lambda_1 u_1(x_1(s)) +\ldots+ \lambda_n u_n(x_n(s))$ s.t. (*)
- For every (i,j,s), $\lambda_i u_i'(x_i^*(s)) = \lambda_j u_j'(x_j^*(s))$

Optimal risk-sharing with CARA

- $u_i(x) = -\exp(-\alpha_i x)$
- $\alpha_i x_i^*(s) = \alpha_j x_j^*(s) + \ln(\lambda_i \alpha_i) - \ln(\lambda_j \alpha_j)$
- i.e. normalized consumption differences are state independent
- Therefore,

$$x_i^{\;*}(s) = \frac{\frac{1}{\alpha_i}}{\frac{1}{\alpha_1}+\cdots+\frac{1}{\alpha_n}} \mathrm{X}(s) + \tau_i$$

where $\tau_1,\cdots,\tau_n$ are deterministic transfers with $\tau_1+\cdots+\tau_n=0$.

- Optimal allocations are obtained by trading the assets.

Application: Insurance

- wealth w and a loss of \$1 with probability p.
- Insurance: pays \$1 in case of loss costs q;
- DM buys λ units of insurance.
- Fact: If $p = q$ (fair premium), then $\lambda = 1$ (full insurance).
  - Expected wealth $w - p$ for all λ.
- Fact: If DM1 buys full insurance, a more risk averse DM2 also buys full insurance.
  - $CE_2(\lambda) \leq CE_1(\lambda) \leq CE_1(1) = CE_2(1)$.

Application: Optimal Portfolio Choice

- With initial wealth w, invest $\alpha \in [0,w]$ in a risky asset that pays a return z per each \$ invested; z has cdf F on $[0,\infty)$.
- $U(\alpha) = \int_0^\infty u(w + \alpha z - \alpha) dF(z)$; concave
- It is optimal to invest $\alpha > 0 \Leftrightarrow E[z] > 1$.
  - $U'(0) = \int_0^\infty u'(w)(z-1)dF(z) = u'(w)(E[z]-1))$.
- If agent with utility u_1 optimally invests α_1, then an agent with more risk averse u_2 (same w) optimally invests $\alpha_2 \leq \alpha_1$.
- DARA ⇒ optimal α increases in w.
- CARA ⇒ optimal α is constant in w.
- CRRA (DRRA) ⇒ optimal α/w is constant (increasing)

Optimal Portfolio Choice – Proof

- $u_2=g(u_1)$; g is concave; $g'(u_1(w)) = 1$.
- $U_i(\alpha) \equiv \int u_i(w+\alpha(z-1))(z-1)\, dF(z)$
- $U_2'(\alpha) - U_1'(\alpha) = \int [u_2'(w+\alpha(z-1)) - u_1'(w+\alpha(z-1))](z-1)dF(z) \leq 0$.
  - $g'(u_1(w+\alpha_1 z-\alpha_1)) < g'(u_1(w)) = 1 \Leftrightarrow z > 1$.
  - $u_2(w+\alpha(z-1)) < u_1(w+\alpha(z-1)) \Leftrightarrow z > 1$.
- $\alpha_2 \leq \alpha_1$

Stochastic Dominance

- Goal: Compare lotteries with minimal assumptions on preferences
- Assume that the support of all payoff distributions is bounded. Support = $[a,b]$.
- Two main concepts:
  - First-order Stochastic Dominance: A payoff distribution is preferred by all monotonic Expected Utility preferences.
  - Second-order Stochastic Dominance: A payoff distribution is preferred by all risk averse EU preferences.

FSD

- DEF: F first-order stochastically dominates G ⇔ for every weakly increasing $u: \mathbb{R} \to \mathbb{R}$, $\int u(x)dF(x) \geq \int u(x)dG(x)$.
- THM: F first-order stochastically dominates G ⇔

  $$F(x) \leq G(x) \text{ for all } x.$$

Proof:

- "Only if:" for $F(x^*) > G(x^*)$, define $u = \mathbf{I}_{\{x>x^*\}}$.
- "If": Assume F and G are strictly increasing and continuous on $[a,b]$.
- Define $y(x) = F^{-1}(G(x))$; $y(x) \geq x$ for all x
- $\int u(y)dF(y) = \int u(y(x))dF(y(x)) = \int u(y(x))dG(x) \geq \int u(x)dG(x)$

MPR and MLR Stochastic Orders

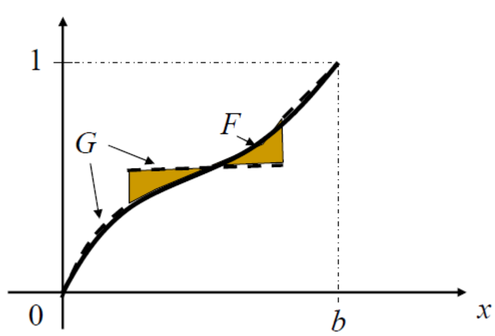
- DEF: F dominates G in the Monotone Probability Ratio (MPR) sense if $k(x) \equiv G(x)/F(x)$ is weakly decreasing in x.
- THM: MPR dominance implies FSD.
- DEF: F dominates G in the Monotone Likelihood Ratio (MLR) sense if $\ell(x) \equiv G'(x)/F'(x)$ is weakly decreasing.
- THM: MLR dominance implies MPR dominance.

SSD

- DEF: F second-order stochastically dominates G ⇔ for every non-decreasing concave u, $\int u(x)dF(x) \geq \int u(x)dG(x)$.
- DEF: G is a mean-preserving spread of F ⇔ $y = x + \varepsilon$ for some $x \sim F$, $y \sim G$, and ε with $E[\varepsilon|x] = 0$.
- THM: Assume: F and G has the same mean. Then, the following are equivalent:
  - F second-order stochastically dominates G.
  - G is a mean-preserving spread of F .
  - $\forall t \geq 0, \int_0^t G(x)dx \geq \int_0^t F(x)dx$.

SSD

- Example: G (dotted) is a mean-preserving spread of F (solid).



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