14.123 Microeconomics III Final Exam Answers
3/22/11
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Instructions. This is an open-book exam. You can use the results in the notes and the answers to the problem sets without proof, but you need to invoke them explicitly. You have 80 minutes. Each question is 25 points. Good Luck!

1. Ann is pregnant. According to the tests so far, there is $p = 1/3000$ chance that the baby has a serious disease. There is a new test that could find out for sure whether the baby has the disease but kills the baby with probability $q = 1/300$. Ann also has the option of aborting the baby after taking the test. Assuming that she is an expected utility maximizer, this question asks you to help her to decide whether to take the test. Here preferences are as follows. She cares only about whether she has a baby and whether that the baby is healthy. Hence, she considers the following consequences:

   (A) a healthy baby;

   (B) a baby with the disease;

   (C) no baby.

   Her utility function is given by $u(A) = 1$, $u(B) = v$ and $u(C) = 0$, where $v \in (-1, 1)$ is known.

   (a) As a function of v, find whether she should take the test.

   Answer: when $v \geq 0$, she would not abort the baby regardless of the test result. Since $u(A) > u(C)$, she does not take the test. When $v \geq 0$, she aborts the baby if and only if the test is positive. Hence, her payoff from taking the test is $(1-p)(1-q)$. Her payoff from not taking the test is $1 - p + pv$. Hence, she takes the test if and only if

   $$v < -q(1-p)/p \cong -q/p = -10.$$

   (b) Suppose now that Ann will learn the test result regardless of whether the baby lives and she cares about how she would feel when she learns the test results. In addition, she now considers the following two consequences:

   (D) baby dies during the test and she learns that the baby was healthy;

   (E) baby dies during the test and she learns that the baby had the disease.

   (Abortion still corresponds to (C).) Assume $1 > u(E) > 0 > v > u(D)$. Find the condition under which she takes the test.

   Answer: The payoff from taking the sets is now $(1-p)(1-q) + qpu(E) + q(1-p)u(D)$. Hence, she takes the test iff

   $$v < -\frac{q}{p}\left[1 - p + pu(E) + (1-p)u(D)\right] \cong \frac{q}{p}\left[u(D) - 1\right].$$

   Since $u(D) < v < 0$ and $q/p = 10$, she does not take the test.

2. Prove or disprove the following.

   (a) Under P1-P2 of Savage, for any events B and C and for any acts f and g, if $f \succeq g$ given B and $f \succeq g$ given C, then $f \succeq g$ given $B \cup C$.

   Answer: False. It is indeed false under expected utility maximization (i.e. P1-P6). Suppose that $p(B \backslash C) = p(B \cap C) = p(C \backslash B) = 1/3$. Suppose $u(g) = 0$ everywhere, while $u(f) = -1$ on $S \backslash (B \cap C)$ and $u(f) = 1.1$ on $B \cap C$. Now, $E[u(f)|B] = E[u(f)|C] = 0.05 > 0 = E[u(g)|B] = E[u(g)|C]$, but $E[u(f)|B \cup C] = E[u(f)] = -0.45 < 0 = E[u(f)|B \cup C]$. (It would habe been true if B and C were disjoint.)

(b) Let lotteries p and q and the vector r be such that $p+r$ and $q+r$ are also lotteries. Under completeness, transitivity and independence axioms of von Neumann and Morngenstern,

$$p \sim q \iff p+r \sim q+r. \tag{1}$$

Answer: True. Define

$$\begin{aligned} \tilde{p} &= p/2 + (p+r)/2 = p + r/2 \\ \tilde{q} &= q/2 + (p+r)/2. \end{aligned}$$

By the independence axiom and completeness,

$$p \sim q \iff \tilde{p} \sim \tilde{q}. \tag{2}$$

Since $\tilde{q} = p/2 + (q+r)/2$, the independence axiom and completeness also imply that

$$p+r \sim q+r \iff \tilde{p} \sim \tilde{q}. \tag{3}$$

By transitivity, (2) and (3) yield (1).

3. Bergson becomes a benevolent dictator. He has n subjects $i = 1, \ldots, n$ with CARA utilities $u_1, \ldots, u_n$, respectively. (Write α_i for the absolute risk aversion of i.) The total wealth in the society, Y, is a function of an unknown state ω and is normally distributed with mean μ and variance σ^2. Bergson can choose any allocation $x = (x_1, \ldots, x_n)$ such that $x_1(\omega) + \cdots + x_n(\omega) \leq Y(\omega)$ at each state ω. His payoff from an allocation x is

$$U(x) = f\left(E\left[u_1(x_1)\right], \ldots, E\left[u_n(x_n)\right]\right),$$

where f is a convex and increasing function.

(a) Assume f is differentiable. What allocation should Bergson chooses to maximize U? (It suffices to find n equations with n unknowns.)

Answer: Since f is increasing, the optimal solution x^* is Pareto optimal. Hence, as we have seen in the class, for each i,

$$x_i^*(\omega) = \frac{1/\alpha_i}{\sum_j 1/\alpha_j} Y(\omega) + \tau_i \tag{4}$$

for some $(\tau_1, \ldots, \tau_n)$ with $\tau_1 + \cdots + \tau_n = 0$. The CE is

$$CE_i(x_i^*) = A\mu/\alpha_i + \tau_i - \frac{1}{2}A^2\sigma^2/\alpha_i,$$

where $A = 1/\sum_j 1/\alpha_j$, and the expected payoff is

$$E\left[u_i(x_i^*)\right] = -\exp\left(-A\mu - \alpha_i\tau_i + \frac{1}{2}A^2\sigma^2\right) \equiv g_i(\tau_i).$$

He can then maximize $f(g_1(\tau_1), \ldots, g_n(\tau_n))$ over $(\tau_1, \ldots, \tau_n)$ subject to $\tau_1 + \cdots + \tau_n = 0$. The first-order conditions yield

$$\alpha_i f_i(g_1(\tau_1), \ldots, g_n(\tau_n)) \exp\left(-A\mu - \alpha_i\tau_i + \frac{1}{2}A^2\sigma^2\right) = \alpha_1 f_1(g_1(\tau_1), \ldots, g_n(\tau_n)) \exp\left(-A\mu - \alpha_1\tau_1 + \frac{1}{2}A^2\sigma^2\right)$$

which simplifies to

$$\alpha_i f_i(g_1(\tau_1), \ldots, g_n(\tau_n)) \exp(-\alpha_i\tau_i) = \alpha_1 f_1(g_1(\tau_1), \ldots, g_n(\tau_n)) \exp(-\alpha_1\tau_1),$$

where f_i is the partial derivative of f with respect to the ith entry.

(b) His good friends Emanuel Kant and John Rawls suggest that he should take $f = \min$. What does he choose under this function? (Find the allocation.)

Answer: Under min, each must get equal expected payoff:

$$A\mu + \alpha_i \tau_i - \frac{1}{2} A^2 \sigma^2 = A\mu + \alpha_1 \tau_1 - \frac{1}{2} A^2 \sigma^2,$$

i.e.

$$\alpha_i \tau_i = \alpha_1 \tau_1.$$

The unique solution to this linear equation system (including $\tau_1 + \cdots + \tau_n = 0$) is

$$\tau_1 = \cdots = \tau_n = 0.$$

Therefore, the solution is

$$x_i^* (\omega) = \frac{1/\alpha_i}{\sum_j 1/\alpha_j} Y (\omega) \qquad (\forall i, \omega).$$

4. Beatrice has initial wealth of w_0 and suffers from quasi-hyperbolic discounting. At any date s, her utility from a consumption stream $x = (x_0, x_1, \ldots)$ is

$$U (x|s) = \ln (x_s) + \beta \sum_{k=1}^{\infty} \delta^k \ln (x_{s+k}),$$

where $\beta, \delta \in (0, 1)$. She gets return of $r > 1$ from her savings so that her wealth at $t + 1$ is $w_{t+1} = r (w_t - x_t)$ if her wealth at t is w_t and she consumes x_t at t.

(a) Find her optimal consumption $x^* = (x_0^*, x_1^*, \ldots)$ for the case of exponential discounting (i.e. $\beta = 1$).

Answer: This is a special case of Problem 3 in PSet 3. Given any w_t, one chooses $x_t = (1 - \delta) w_t$, yielding

$$x_t^* = (1 - \delta) w_0 (\delta r)^t.$$

(b) Find a sophisticated-optimal consumption strategy for her in which the self at any given date s consumes γw_s. Compute the constant γ and briefly verify that this is indeed a subgame-perfect equilibrium of the multi-agent game.

Answer: Given γ, the present value of wealth w is

$$V (w|\gamma) = \sum_{t=0}^{\infty} \delta^t \ln (x_t) = \sum_{t=0}^{\infty} \delta^t \ln \left(\gamma (r (1 - \gamma))^t w \right) = \frac{1}{1 - \delta} \ln (w) + V (1|\gamma),$$

where

$$V (1|\gamma) = \sum_{t=0}^{\infty} \delta^t \ln \left(\gamma (r (1 - \gamma))^t \right) = \frac{1}{1 - \delta} \ln \gamma + \frac{\delta}{(1 - \delta)^2} \ln (r (1 - \gamma)).$$

For any wealth level w_t, the self at t maximizes

$$\ln (x_t) + \beta \delta V (r (w_t - x_t)),$$

choosing

$$x_t = \frac{1}{1 + \beta\delta / (1 - \delta)} w_t.$$

Hence,

$$\gamma = \frac{1}{1 + \beta\delta / (1 - \delta)}.$$

Since the self is indeed choosing $x_t = \gamma w_t$ as a best response to the future selves, this is indeed a subgame-perfect equilibrium.

(c) For $\beta < 1$, find conditions under which there is a sophisticated optimal consumption strategy under which she consumes according to x^* on the path of play. (Do not lose time to simplify your expressions.)

[Hint: Use the strategy in part (b) as the punishment for deviations from the desired path.]

Answer: Consider the following strategy. At any given history, as self at date s consumes x_s^* if the self at t has consumed x_t^* for each $t < s$, and the self at s consumes γw_t otherwise. To check that this is indeed a SPE of the multi-agent game it suffices to check that the self does not deviate if all previous selves consumed according to x^*. (This is because after deviation, we play according to the equilibrium in part (b).) Now at any such history, if the self at s follows the strategy by consuming $(1-\delta) w_s$, all the future selves will consume according to that rule, yielding the expected payoff of

$$U(w_s, 1-\delta) = \ln((1-\delta) w_s) + \beta\delta V(r\delta w_s | 1-\delta).$$

If she were to deviate, then the most profitable deviation is consuming γw_s (by part (b)), yielding the expected payoff of

$$U(w_s, \gamma) = \ln(\gamma w_s) + \beta\delta V(r(1-\gamma) w_s | 1-\delta).$$

She does not have an incentive to deviate iff

$$U(w_s, 1-\delta) \geq U(w_s, \gamma).$$

This simplifies to the same condition for all selves:

$$\ln(1-\delta) + \beta\delta\left[\frac{1}{1-\delta}\ln(r\delta) + V(1|1-\delta)\right] \geq \ln(\gamma) + \beta\delta\left[\frac{1}{1-\delta}\ln(r(1-\gamma)) + V(1|\gamma)\right].$$

In other words, all selves have the same preference regarding the exponential and hyperbolic plans. The new strategy is sophisticated optimal iff and only if the exponential plan in (a) is better than the hyperbolic plan in (b).

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