πEcon 14.04 Fall 2006
Solutions: PS1

1. (a) Setting up the optimization problem we have:

$$L(p_1, p_2, m) = u(x_1, x_2) + \lambda(m - p_1x_1 - p_2x_2)$$

The FOC yield:

$$\begin{aligned} (1)\frac{\partial L}{\partial x_1} &= \frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda p_1 = 0 \\ (2)\frac{\partial L}{\partial x_2} &= \frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda p_2 = 0 \end{aligned}$$

Dividing (1) from (2) yields:

$$\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = \frac{p_1}{p_2}$$

(b) Setting up the optimization problem we have:

$$L(p_1, p_2, m) = v(u(x_1, x_2)) + \lambda(m - p_1x_1 - p_2x_2)$$

The FOC yield:

$$\begin{aligned} (1)\frac{\partial L}{\partial x_1} &= v'(u(x_1, x_2))\frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda p_1 = 0 \\ (2)\frac{\partial L}{\partial x_2} &= v'(u(x_1, x_2))\frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda p_2 = 0 \end{aligned}$$

Dividing (1) from (2) yields:

$$\frac{v'(u(x_1, x_2))}{v'(u(x_1, x_2))}\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = \frac{p_1}{p_2}$$

(c) Notice that as long as $v'(u(x_1, x_2))$ is never equal to zero, problem b problem has exactly the same solution as the problem in part a. Thus monotonic transformations do not alter the marshallian demand functions

(a) As we did in recitation, we take the log of the function and show that this is monotonic and convex:

Monotinicity:

$$a\ln(x + \varepsilon_x) + \ln(y + \varepsilon_y) > a\ln(x) + \ln(y)\ \forall \varepsilon_x, \varepsilon_y > 0$$

Convexity:

$$M = \begin{bmatrix} -\frac{a}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{bmatrix} \rightarrow \gamma \quad \beta \begin{bmatrix} -\frac{a}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{bmatrix} \begin{matrix} \gamma \\ \beta \end{matrix} = -\frac{\gamma^2 a}{x^2} - \frac{\beta^2}{y^2}$$

Since $z'Mz < 0$, the utility curve is quasiconcave which imply that the indifference curves are convex.

(b)

$$K(\alpha, p_1, p_2) = a \ln x + \ln y + \lambda_B(m - x - py) + \lambda_x x + \lambda_y y$$

Taking FOC:

$$\begin{aligned}
(1)\frac{\partial K}{\partial x_1} &= \frac{\alpha}{x} - \lambda_B + \lambda_x = 0 \\
(2)\frac{\partial K}{\partial x_2} &= \frac{1}{y} - \lambda_B p + \lambda_y = 0 \\
(3)\frac{\partial K}{\partial \lambda_1} &= m - x - py = 0 \\
(4)\frac{\partial K}{\partial \lambda_2} &= x \geq 0, \lambda_x \geq 0, \lambda_x x = 0 \\
(5)\frac{\partial K}{\partial \lambda_2} &= y \geq 0, \lambda_y \geq 0, \lambda_y y = 0
\end{aligned}$$

Assume $\lambda_x > 0 \rightarrow x = 0 \rightarrow \lambda_B = \infty \rightarrow y = 0, Contradicts\ (3)$
Assume $\lambda_y > 0 \rightarrow y = 0 \rightarrow \lambda_B = \infty \rightarrow x = 0, Contradicts\ (3)$

Setting $\lambda_x, \lambda_y = 0$, we can divide (1) by (2) to find: $\frac{ay}{x} = \frac{1}{p} \rightarrow x = apy$

Plugging these into (3) yeilds:

$$\begin{aligned}
x &= \frac{am}{1+a} \\
y &= \frac{m}{(1+a)p} \\
v(a, p, m) &= \left(\frac{am}{1+a}\right)^a \frac{m}{(1+a)p}
\end{aligned}$$

Note that this problem would be very easy if we ignore the inequality constraints. See problem 4

2.

$$K(\alpha, p_1, p_2) = \ln x + y + \lambda_B(10 - 2x - y) + \lambda_x x + \lambda_y y$$

FOC:

$$\begin{aligned}
(1)\frac{\partial K}{\partial x} &: \frac{1}{x} - 2\lambda_B + \lambda_x = 0 \\
(2)\frac{\partial K}{\partial x} &: 1 - \lambda_B + \lambda_y = 0 \\
(3)\frac{\partial K}{\partial \lambda_B} &: 2x + y = 10 \\
(4)\frac{\partial K}{\partial \lambda_x} &: x \geq 0, \lambda_x \geq 0, \lambda_x x = 0 \\
(5)\frac{\partial K}{\partial \lambda_y} &: y \geq 0, \lambda_y \geq 0, \lambda_y y = 0
\end{aligned}$$

Assume $\lambda_x = 0 \rightarrow x = 0 \rightarrow \lambda_B = \infty \rightarrow \lambda_y > 0 \rightarrow y = 0 : Contradition\ w/\ (3)$

Dividing (1) by (2) we have:

$$\frac{1}{x + \lambda_y} = 2 \rightarrow x = \frac{1}{2} - \lambda_y$$

$x = 1/2$ unless y = 0. Thus from (3):

$$\begin{aligned} x &= \frac{1}{2} \\ y &= 9 \end{aligned}$$

(a) $lim_{y \rightarrow 0} MRS_{xy}(x, y) = lim_{y \rightarrow 0} \frac{x}{y} = \infty$
$lim_{x \rightarrow 0} MRS_{yx}(x, y) = lim_{x \rightarrow 0} \frac{y}{x} = \infty$

(b) $lim_{y \rightarrow 0} MRS_{xy}(x, y) = lim_{y \rightarrow 0} 1 = 1$
$lim_{x \rightarrow 0} MRS_{yx}(x, y) = lim_{x \rightarrow 0} 1 = 1$

(c) $lim_{y \rightarrow 0} MRS_{xy}(x, y) = lim_{y \rightarrow 0} x = x$
$lim_{x \rightarrow 0} MRS_{yx}(x, y) = lim_{x \rightarrow 0} \frac{1}{x} = \infty$

Notice that utilities where the constraints may bind are those where the MRS does not go off to infinity. If you think about it for a bit, this should make sense. The FOC for the standard utiltiy function sets MRS $= \frac{p_1}{p_2}$. If the MRS goes to infinity this says that for any price vector, there is a place on the utility curve in R^2_+ that has the same slope.