Q1. Fig. **??** depicts a representation of $f(x)$ by the values f_i at points x_i with variable grid-spacing $\Delta x_{i+\frac{1}{2}}$ such that $\Delta x_{i+\frac{1}{2}} = x_{i+1} - x_i$. Using Taylor series, derive a second order finite-difference approximation to $\partial_x f$ at the location x_i using a three point stencil involving f_{i-1}, f_i and f_{i+1}.
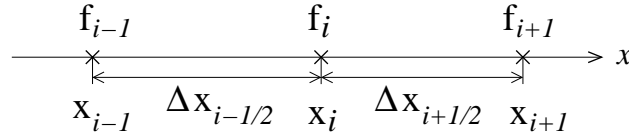

Figure 1: Three point stencil on a variable resolution grid.

Q2.a Assume a regular grid with index i such that $x_i = i\Delta x$ and function values $f_i = f(x_i)$. The expression

$$\frac{f_{i+1} - f_i}{\Delta x} = \partial_x f + O(\Delta x) \tag{1}$$

is a "side difference" approximation of order $O(\Delta x)$ for $\partial_x f$ evaluated at the x_i point.

i) Derive the actual truncation error terms out to order Δx^2.

ii) Derive an approximation for $\partial_{xx} f$ evaluated at the same point, x_i, that uses the values f_{i-1}, f_i and f_{i+1}.

iii) Substitute your approximation to $\partial_{xx} f$ from (ii) into the leading truncation term in equation (**??**). What is the leading order truncation term and resulting scheme?

Q2.b On a regular grid (spacing Δx), the second order, centered approximation of the first derivative at a grid-node, x_i, is

$$\frac{f_{i+1} - f_{i-1}}{2\Delta x} = \partial_x f + O(\Delta x^2). \tag{2}$$

i) Derive the leading order truncation error in (**??**).

ii) Using the stencil $(f_{i-2}, f_{i-1}, f_i, f_{i+1})$ write a finite difference approximation for $\partial_{xxx} f$.

iii) Substitute your approximation from (ii) into the leading order error term of the approximation to $\partial_x f$ in equation (**??**) and re-evaluate the truncation error of the resulting expression. This is an approximation for $\partial_x f$ of what order?

Q2.c Do you see the pattern behind the methods you used in Q2.a and b. Briefly explain, how you would derive an $O(\Delta x^n)$ approximation to a finite difference expression if you were given an $O(\Delta x^{n-1})$ finite difference expression.

Q3. Assume a regular grid as in Q2.a.

i) Derive a second order expression of a similar form as (**??**) but using only the values f_{i-2} and f_{i+2}.

ii) Linearly combine your approximations from (i) and the approximation in equation (**??**) to yield a $O(\Delta x^4)$ approximation for $\partial_x f$ at x_i.

Q4. The linear advection problem

$$\partial_t \theta + u \partial_x \theta = 0 \tag{3}$$

can be written in the flux form

$$\partial_t \theta + \partial_x F = 0 \quad \text{where} \quad F = u\theta.$$

Here F is an advective flux and u is the constant flow speed. We will assume that $u > 0$ for this discussion. We wish to discretize the flux-form as

$$\partial_t \theta + \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x} = 0 \quad \text{or} \quad \partial_t \theta + \frac{1}{\Delta x}\delta_i F = 0 \tag{4}$$

on the regular grid of Q2.a. We will assume perfect derivatives in time.

i) Write down an $O(\Delta x^3)$ approximation for $F_{i+\frac{1}{2}}$ in terms of θ_{i-1}, θ_i, θ_{i+1}. This is known as the third-order, upwind-biased flux because $u > 0$.

ii) Substitute your $O(\Delta x^3)$ expressions for $F_{i+\frac{1}{2}}$ and $F_{i-\frac{1}{2}}$ into equation (**??**) and derive the leading order truncation error. We substituted a third order approximation of the flux so why is the approximation to the governing equation not third order?

iii) Write down the $O(\Delta x^3)$ finite difference approximate to equation (**??**) evaluated at x_i using the stencil (θ_{i-2}, θ_{i-1}, θ_i, θ_{i+1}).

iv) The approximation to the term $c\partial_x \theta$ you used in (iii) takes the form

$$a\theta_{i-2} + b\theta_{i-1} + c\theta_i + d\theta_{i+1}$$

or

$$\begin{bmatrix} a & b & c & d \end{bmatrix}$$

using stencil short-hand. Let us write this stencil as the difference between two stencils of the same but shifted form:

$$[a \ b \ c \ d] = [0 \ \alpha \ \beta \ \gamma] - [\alpha \ \beta \ \gamma \ 0].$$

Find α, β and γ. Deduce an approximation for the the flux $F_{i+\frac{1}{2}}$ that yields an $O(\Delta x^3)$ approximation to the linear advection problem (**??**).

Q5 (MATLAB)
    Build a model that integrates the linear advection equation

$$\partial_t \theta + u \partial_x \theta = 0$$

in the periodic domain $0 \leq x \leq 1$ which is divided into N even intervals, $\Delta x = 1/N$. $u = 1$ is a constant flow. Use the flux-form and march the system forward as follows:

$$\theta^{(n+1)} = \theta^{(n)} - \frac{\Delta t}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$$

where the superscript (n) is the time-level. *Note: this is the forward scheme and as a rough rule will not work for many advection schemes - but it will work for the schemes you will try here.* For this problem set, use $\Delta t = \frac{1}{100}\frac{\Delta x}{u}$. The initial conditions will be a unit amplitude cosine-bell given by:

$$\theta(x, t = 0) = \frac{1}{2}\left(1 + \cos\left[\pi \min\left[1, |4(x - \frac{1}{2})|\right]\right]\right)$$

Start by using the "upwind" scheme which corresponds to a flux given by

$$F_{i+\frac{1}{2}} = u\theta_i.$$

You will know you have debugged your model when the range of values of your solution are within those of the initial conditions (using the upwind scheme and for the given parameters).

    i) Plot the initial conditions and solutions at time $t = 1$ for $N = 10, 20, 40, 80$.

    ii) At each resolution, measure the l_1, l_2 and l_∞ norms. Plot them as a function of grid-spacing and measure the power dependence of the l_2 curve on Δx.

    iii) Repeat (i) and (ii) using the two forms of "third" order flux in Q4. Which is more accurate?

    iv) Why is the dependence not what you would expect, given all the effort you put into deriving the truncation errors in the previous questions?