## 6. Discrete Fourier Analysis

The expression (4.2) shows that the Fourier transform of a discrete process is a function of exp $(-2\pi is)$ and hence is periodic with period 1 (or $1/\Delta t$ for general $\Delta t$). A finite data length means that all of the information about it is contained in its values at the special frequencies $s_n = n/T$. If we define

$$z = e^{-2\pi is\Delta t} \tag{6.1}$$

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the Fourier transform is

$$\hat{x}(s) = \sum_{m=-T/2}^{T/2} x_m z^m \tag{6.2}$$

We will write this, somewhat inconsistently interchangeably, as $\hat{x}(s), \hat{x}\left(e^{-2\pi is}\right), \hat{x}(z)$ where the two latter functions are identical; $\hat{x}(s)$ is clearly not the same function as $\hat{x}\left(e^{-2\pi is}\right)$, but the context should make clear what is intended. Notice that $\hat{x}(z)$ is just a polynomial in $z$, with negative powers of $z$ multiplying $x_n$ at negative times. That a Fourier transform (or series–which differs only by a constant normalization) is a polynomial in $\exp(-2\pi is)$ proves to be a simple, but powerful idea.

*Definition.* We will use interchangeably the terminology "sequence", "series" and "time series", for the discrete function $x_m$, whether it is discrete by definition, or is a sampled continuous function. Any subscript implies a discrete value.

Consider for example, what happens if we multiply the Fourier transforms of $x_m, y_m$ :

$$\hat{x}(z)\,\hat{y}(z) = \left(\sum_{m=-T/2}^{T/2} x_m z^m\right)\left(\sum_{k=-T/2}^{T/2} y_k z^k\right) = \sum_k \left(\sum_m x_m y_{k-m}\right) z^k = \hat{h}(z)\,. \tag{6.3}$$

That is to say, the product of the two Fourier transforms is the Fourier transform of a new time series,

$$h_k = \sum_{m=-\infty}^{\infty} x_m y_{k-m}, \tag{6.4}$$

which is the rule for polynomial multiplication, and is a discrete generalization of convolution. The infinite limits are a convenience—most often one or both time series vanishes beyond a finite value.

More generally, the algebra of discrete Fourier transforms is the algebra of polynomials. We could ignore the idea of a Fourier transform altogether and simply define a transform which associates any sequence $\{x_m\}$ with the corresponding polynomial (6.2), or formally

$$\{x_m\} \longleftrightarrow \mathcal{Z}(x_m) \equiv \sum_{m=-T/2}^{T/2} x_m z^m \tag{6.5}$$

The operation of transforming a discrete sequence into a polynomial is called a $z-$transform. The $z-$transform coincides with the Fourier transform on the unit circle $|z| = 1$. If we regard $z$ as a general complex variate, as the symbol is meant to suggest, we have at our disposal the entire subject of complex functions to manipulate the Fourier transforms, as long as the corresponding functions are finite on the unit circle. Fig. 8 shows how the complex $s-$plane, maps into the complex $z-$plane, the real line in the former, mapping into the unit circle, with the upper half-$s-$plane becoming the interior of $|z| = 1$

There are many powerful results. One simple type is that any function analytic on the unit circle corresponds to a Fourier transform of a sequence. For example, suppose

$$\hat{x}(z) = Ae^{az} \tag{6.6}$$

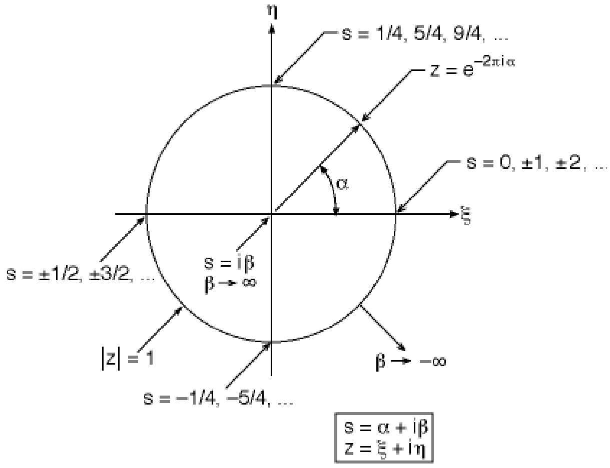


FIGURE 8. Relationships between the complex $z$ and $s$ planes.

Because $\exp(az)$ is analytic everywhere for $|z| < \infty$, it has a convergent Taylor Series on the unit circle

$$\hat{x}(z) = A\left(1 + az + a^2\frac{z^2}{2!} + ....\right) \tag{6.7}$$

and hence $x_0 = A, x_1 = Aa, x_2 = Aa^2/2!, ...$Note that $x_m = 0, m < 0$. Such a sequence, vanishing for negative $m$, is known as a "causal" one.

*Exercise.* Of what sequence is $A\sin(bz)$ the $z-$transform? What is the Fourier Series? How about,

$$\hat{x}(z) = \frac{1}{(1-az)(1-bz)}, a > 1, b < 1? \tag{6.8}$$

This formalism permits us to define a "convolution inverse". That is, given a sequence, $x_m$, can we find a sequence, $b_m$, such that the discrete convolution

$$\sum_k b_k x_{m-k} = \sum_k b_{m-k} x_k = \delta_{m0} \tag{6.9}$$

where $\delta_{m0}$ is the Kronecker delta (the discrete analogue of the Dirac $\delta$)? To find $b_m$, take the $z-$transform of both sides, noting that $\mathcal{Z}(\delta_{m0}) = 1$, and we have

$$\hat{b}(z)\hat{x}(z) = 1 \tag{6.10}$$

or

$$\hat{b}(z) = \frac{1}{\hat{x}(z)} \tag{6.11}$$

But since $\hat{x}(z)$ is just a polynomial, we can find $\hat{b}(z)$ by simple polynomial division.

*Example.* Let $x_m = 0, m < 0, x_0 = 1, x_1 = 1/2, x_2 = 1/4, x_m = 1/8, ...$ What is its convolution inverse? $\mathcal{Z}(x_m) = 1 + z/2 + z^2/4 + z^3/8 + ....$ So

$$\hat{x}(z) = 1 + z/2 + z^2/4 + z^3/8 + .... = \frac{1}{1-(1/2)z} \tag{6.12}$$

so $\hat{b}(z) = 1 - (1/2)z$, with $b_0 = 1, b_1 = -1/2, b_m = 0$, otherwise.

*Exercise.* Confirm by direct convolution that the above $b_m$ is indeed the convolution inverse of $x_m$.

This idea leads to the extremely important field of "deconvolution". Define

$$h_m = \sum_{n=-\infty}^{\infty} f_n g_{m-n} = \sum_{n=-\infty}^{\infty} g_n f_{m-n}. \tag{6.13}$$

Define $g_m = 0, m < 0$; that is, $g_m$ is causal. Then the second equality in (6.13) is

$$h_m = \sum_{n=0}^{\infty} g_n f_{m-n}, \tag{6.14}$$

or writing it out,

$$h_m = g_0 f_m + g_1 f_{m-1} + g_2 f_{m-2} + .... \tag{6.15}$$

If time $t = m$ is regarded as the "present", then $g_n$ operates only upon the present and earlier (the past) values of $f_k$; no future values of $f_m$ are required. Causal sequences $g_n$ appear, e.g., when one passes a signal, $f_k$ through a linear system which does not respond to the input before it occurs, that is the system is causal. Indeed, the notation $g_n$ has been used to suggest a Green function. So-called real time filters are always of this form; they cannot operate on observations which have not yet occurred.

In general, whether a $z-$transform requires positive, or negative powers of $z$ (or both) depends only upon the location of the singularities of the function relative to the unit circle. If there are singularities in $|z| < 1$, a Laurent series is required for convergence on $|z| = 1$; if all of the singularities occur for $|z| > 1$, a Taylor Series will be convergent and the function will be causal. If both types of singularities are present, a Taylor-Laurent Series is required and the sequence cannot be causal. When singularities exist on the unit circle itself, as with Fourier transforms with singularities on the real $s$-axis one must decide through a limiting process what the physics are.

Consider the problem of deconvolving $h_m$ in (6.15) from a knowledge of $g_n$ and $h_m$, that is one seeks $f_k$. From the convolution theorem,

$$\hat{f}(z) = \frac{\hat{h}(z)}{\hat{g}(z)} = \hat{h}(z)\,\hat{a}(z). \tag{6.16}$$

Could one find $f_k$ given only the past and present values of $h_m$? Evidently, that requires a filter $\hat{a}(z)$ which is also causal. Thus it cannot have any poles inside $|z| < 1$. The poles of $\hat{a}(z)$ are evidently the zeros of $\hat{g}(z)$ and so the latter cannot have any zeros inside $|z| < 1$. Because $\hat{g}(z)$ is itself causal, if it is to have a (stable) causal inverse, it cannot have either poles or zeros inside the unit circle. Such a sequence $g_m$ is called "minimum phase" and has a number of interesting and useful properties (see e.g., Claerbout, 1985).

As one example, consider that it is possible to show that for *any* stationary, stochastic process, $x_n$, that one can always write it as

$$x_n = \sum_{k=0}^{\infty} a_n \theta_{n-k},\ a_0 = 1$$

where $a_n$ is minimum phase and $\theta_n$ is white noise, with $< \theta_n^2 >= \sigma_\theta^2$. Let $n$ be the present time. Then one time-step in the future, one has

$$x_{n+1} = \theta_{n+1} + \sum_{k=1}^{\infty} a_n \theta_{n-k}.$$

Now at time $n$, $\theta_{n+1}$ is completely unpredictable. Thus the best possible prediction is

$$\tilde{x}_{n+1} = 0 + \sum_{k=1}^{\infty} a_n \theta_{n-k}. \tag{6.17}$$

with expected error,

$$< (\tilde{x}_{n+1} - x_{n+1})^2 >=< \theta_n^2 >= \sigma_\theta^2.$$

It is possible to show that this prediction, given $a_n$, is the best possible one; no other predictor can have a smaller error than that given by the minimum phase operator $a_n$. If one wishes a prediction $q$ steps into the future, then it follows immediately that

$$\tilde{x}_{n+q} = \sum_{k=q}^{\infty} a_k \theta_{n+q-k},$$

$$< (\tilde{x}_{n+q} - x_{n+q})^2 >= \sigma_\theta^2 \sum_{k=0}^{q} a_k^2$$

which sensibly, has a monotonic growth with $q$. Notice that $\theta_k$ is determinable from $x_n$ and its *past* values only, as the minimum phase property of $a_k$ guarantees the existence of the convolution inverse filter, $b_k$, such that,

$$\theta_n = \sum_{k=0}^{\infty} b_k x_{n-k},\ b_0 = 1.$$

*Exercise.* Consider a $z-$transform

$$\hat{h}(z) = \frac{1}{1-az} \tag{6.18}$$

and find the corresponding sequence $h_m$ when $a \to 1$ from above, and from below.

It is helpful, sometimes, to have an inverse transform operation which is more formal than saying "read off the corresponding coefficient of $z^m$). The inverse operator $\mathcal{Z}^{-1}$ is just the Cauchy Residue Theorem

$$x_m = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\hat{x}(z)}{z^{m+1}} dz. \tag{6.19}$$

We leave all of the details to the textbooks (see especially, Claerbout, 1985).

The discrete analogue of cross-correlation involves two sequences $x_m$, $y_m$ in the form

$$r_\tau = \sum_{n=-\infty}^{\infty} x_n y_{n+\tau} \tag{6.20}$$

which is readily shown to be the convolution of $y_m$ with the *time-reverse* of $x_n$. Thus by the discrete time-reversal theorem,

$$\mathcal{F}(r_\tau) = \hat{r}(s) = \hat{x}(s)^* \hat{y}(s). \tag{6.21}$$

Equivalently,

$$\hat{r}(z) = \hat{x}\left(\frac{1}{z}\right)\hat{y}(z). \tag{6.22}$$

$\hat{r}\left(z = e^{-2\pi i s}\right) = \Phi_{xy}(s)$ is known as the cross-power spectrum of $x_n, y_n$.

If $x_n = y_n$, we have discrete autocorrelation, and

$$\hat{r}(z) = \hat{x}\left(\frac{1}{z}\right)\hat{x}(z). \tag{6.23}$$

Notice that wherever $\hat{x}(z)$ has poles and zeros, $\hat{x}(1/z)$ will have corresponding zeros and poles. $\hat{r}\left(z = e^{-2\pi i s}\right) = \Phi_{xx}(s)$ is known as the power spectrum of $x_n$. Given any $\hat{r}(z)$, the so-called spectral factorization problem consists of finding two factors $\hat{x}(z)$, $\hat{x}(1/z)$ the first of which has all poles and zeros outside $|z| = 1$, and the second having the corresponding zeros and poles inside. The corresponding $x_m$ would be minimum phase.

*Example.* Let $x_0 = 1, x_1 = 1/2, x_n = 0, n \neq 0, 1$. Then $\hat{x}(z) = 1 + z/2$, $\hat{x}(1/z) = 1 + 1/(2z)$, $\hat{r}(z) = (1 + 1/(2z))(1 + z/2) = 5/4 + 1/2(z + 1/z)$. Hence $\Phi_{xx}(s) = 5/4 + \cos(2\pi s)$.

**Convolution as a Matrix Operation**

Suppose $f_n, g_n$ are both causal sequences. Then their convolution is

$$h_m = \sum_{n=0}^{\infty} g_n f_{m-n} \tag{6.24}$$

or writing it out,

$$\begin{aligned} h_0 &= f_0 g_0 \\ h_1 &= f_0 g_1 + f_1 g_0 \\ h_2 &= f_0 g_2 + f_1 g_1 + f_2 g_0 \\ &\ldots \end{aligned} \tag{6.25}$$

which we can write in vector matrix form as

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ . \end{bmatrix} = \left\{ \begin{matrix} g_0 & 0 & 0 & 0 & . & 0 \\ g_1 & g_0 & 0 & 0 & . & 0 \\ g_2 & g_1 & g_0 & 0 & . & 0 \\ . & . & . & . & . & . \end{matrix} \right\} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ . \end{bmatrix},$$

or because convolution commutes, alternatively as

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ . \end{bmatrix} = \left\{ \begin{matrix} f_0 & 0 & 0 & 0 & . & 0 \\ f_1 & f_0 & 0 & 0 & . & 0 \\ f_2 & f_1 & f_0 & 0 & . & 0 \\ . & . & . & . & . & . \end{matrix} \right\} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ . \end{bmatrix},$$

which can be written compactly as

$$\mathbf{h} = \mathbf{G}\mathbf{f} = \mathbf{F}\mathbf{g}$$

where the notation should be obvious. Deconvolution then becomes, e.g.,

$$\mathbf{f} = \mathbf{G}^{-1}\mathbf{h},$$

if the matrix inverse exists. These forms allow one to connect convolution, deconvolution and signal processing generally to the matrix/vector tools discussed, e.g., in Wunsch (1996). Notice that causality was not actually required to write convolution as a matrix operation; it was merely convenient.

**Starting in Discrete Space**

One need not begin the discussion of Fourier transforms in continuous time, but can begin directly with a discrete time series. Note that some processes are by nature discrete (e.g., population of a list of cities; stock market prices at closing-time each day) and there need not be an underlying continuous process. But whether the process is discrete, or has been discretized, the resulting Fourier transform is then periodic in frequency space. If the duration of the record is finite (and it could be physically of limited lifetime, not just bounded by the observation duration; for example, the width of the Atlantic Ocean is finite and limits the wavenumber resolution of any analysis), then the resulting Fourier transform need be computed only at a finite, countable number of points. Because the Fourier sines and cosines (or complex exponentials) have the somewhat remarkable property of being exactly orthogonal not only when integrated over the record length, but also of being exactly orthogonal when *summed* over the same interval, one can do the entire analysis in discrete form.

Following the clear discussion in Hamming (1973, p. 510), let us for variety work with the real sines and cosines. The development is slightly simpler if the number of data points, $N$, is even, and we confine the discussion to that (if the number of data points is in practice odd, one can modify what follows, or simply add a zero data point, or drop the last data point, to reduce to the even number case). Define $T = N\Delta t$ (notice that the basic time duration is not $(N-1)\,\Delta t$ which is the true data duration, but has one extra time step. Then the sines and cosines have the following orthogonality properties:

$$\sum_{p=0}^{N-1}\cos\left(\frac{2\pi k}{T}p\Delta t\right)\cos\left(\frac{2\pi m}{T}p\Delta t\right)=\begin{cases}0 & k\neq m\\ N/2\ , & k=m\neq 0,N/2\\ N & k=m=0,N/2\end{cases} \tag{6.26}$$

$$\sum_{p=0}^{N-1}\sin\left(\frac{2\pi k}{T}p\Delta t\right)\sin\left(\frac{2\pi m}{T}p\Delta t\right)=\begin{cases}0 & k\neq m\\ N/2\ , & k=m\neq 0,N/2\end{cases} \tag{6.27}$$

$$\sum_{p=0}^{N-1}\cos\left(\frac{2\pi k}{T}p\Delta t\right)\sin\left(\frac{2\pi m}{T}p\Delta t\right)=0. \tag{6.28}$$

Zero frequency, and the Nyquist frequency, are evidently special cases. Using these orthogonality properties the expansion of an arbitrary sequence at data points $m\Delta t$ proves to be:

$$x_m=\frac{a_0}{2}+\sum_{k=1}^{N/2-1}a_k\cos\left(\frac{2\pi km\Delta t}{T}\right)+\sum_{k=1}^{N/2-1}b_k\sin\left(\frac{2\pi km\Delta t}{T}\right)+\frac{a_{N/2}}{2}\cos\left(\frac{2\pi Nm\Delta t}{2T}\right), \tag{6.29}$$

where

$$a_k=\frac{2}{N}\sum_{p=0}^{N-1}x_p\cos\left(\frac{2\pi kp\Delta t}{T}\right),k=0,...,N/2 \tag{6.30}$$

$$b_k=\frac{2}{N}\sum_{p=0}^{N-1}x_p\sin\left(\frac{2\pi kp\Delta t}{T}\right),k=1,...N/2-1. \tag{6.31}$$

The expression (6.29) separates the 0 and Nyquist frequencies and whose sine coefficients always vanish; often for notational simplicity, we will assume that $a_0, a_N$ vanish (removal of the mean from a time series is almost always the first step in any case, and if there is significant amplitude at the Nyquist frequency, one probably has significant aliasing going on.). Notice that as expected, it requires $N/2+1$ values of $a_k$ and $N/2-1$ values of $b_k$ for a total of $N$ numbers in the frequency domain, the same total numbers as in the time-domain.

*Exercise.* Write a computer code to implement (6.30,6.31) directly. Show numerically that you can recover an arbitrary sequence $x_{p.}$

The complex form of the Fourier *series,* would be

$$x_m=\sum_{k=-N/2}^{N/2}\alpha_k e^{2\pi ikm\Delta t/T} \tag{6.32}$$

$$\alpha_k=\frac{1}{N}\sum_{p=0}^{N-1}x_p e^{-2\pi ikp\Delta t/T}. \tag{6.33}$$

This form follows from multiplying (6.32) by $\exp\left(-2\pi imr\Delta t/T\right)$, summing over $m$ and noting

$$\sum_{m=0}^{N-1}e^{(k-r)2\pi im\Delta t/T}=\begin{cases}N, \quad k=r\\ \left(1-e^{(2\pi i(k-r))}\right)/\left(1-e^{(2\pi i(k-r)/N)}\right)=0, \quad k\neq r\end{cases}. \tag{6.34}$$

The last expression follows from the finite summation formula for geometric series,

$$\sum_{j=0}^{N-1} ar^j = a\frac{1-r^N}{1-r}. \tag{6.35}$$

The Parseval Relationship becomes

$$\frac{1}{N}\sum_{m=0}^{N-1} x_m^2 = \sum_{k=-N/2}^{N/2} |\alpha_k|^2 . \tag{6.36}$$

The number of complex coefficients $\alpha_k$ appears to involve $2(N/2)+1 = N+1$ complex numbers, or $2N+2$ values, while the $x_m$ are only $N$ real numbers. But it follows immediately that if $x_m$ are real, that $\alpha_{-k} = \alpha_k^*$, so that there is no new information in the negative index values, and $\alpha_0, \alpha_{N/2} = \alpha_{-N/2}$ are both real so that the number of Fourier series values is identical. Note that the Fourier transform values,$\hat{x}(s_n)$ at the special frequencies $s_n = 2\pi n/T$, are

$$\hat{x}(s_n) = N\alpha_n, \tag{6.37}$$

so that the Parseval relationship is modified to

$$\sum_{m=0}^{N-1} x_m^2 = \frac{1}{N}\sum_{k=-N/2}^{N/2} |\hat{x}(s_n)|^2 . \tag{6.38}$$

To avoid negative indexing issues, many software packages redefine the baseband to lie in the positive range $0 \le k \le N$ with the negative frequencies appearing after the positive frequencies (see, e.g., Press et al., 1992, p. 497). Supposing that we do this, the complex Fourier transform can be written in vector/matrix form. Let $z_n = e^{-2\pi i s_n t}$, then

$$\begin{bmatrix} \hat{x}(s_0) \\ \hat{x}(s_1) \\ \hat{x}(s_2) \\ . \\ \hat{x}(s_m) \\ . \end{bmatrix} = \left\{ \begin{matrix} 1 & 1 & 1 & . & 1 & . \\ 1 & z_1^1 & z_1^2 & . & z_1^N & . \\ 1 & z_2^1 & z_2^2 & . & z_2^N & . \\ . & . & . & . & . & . \\ 1 & z_m^1 & z_m^2 & . & z_m^N & . \\ . & . & . & . & . & . \end{matrix} \right\} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ . \\ x_q \\ . \end{bmatrix}. \tag{6.39}$$

or,

$$\hat{\mathbf{x}} = \mathbf{B}\mathbf{x}, \tag{6.40}$$

The inverse transform is thus just

$$\mathbf{x} = \mathbf{B}^{-1}\hat{\mathbf{x}}, \tag{6.41}$$

and the entire numerical operation can be thought of as a set of simultaneous equations, e.g., (6.41), for a set of unknowns $\hat{\mathbf{x}}$.

The relationship between the complex and the real forms of the Fourier series is found simply. Let $\alpha_n = c_n + id_n$, then for real $x_m$, (6.32) is,

$$x_m = \sum_{n=0}^{N/2} (c_n + id_n)(\cos(2\pi nm/T) + i\sin(2\pi nm/T)) + (c_n - id_n)(\cos(2\pi nm/T) - i\sin(2\pi nm/T))$$

$$= \sum_{n=0}^{N/2} \{2c_n \cos(2\pi nm/T) - 2d_n \sin(2\pi nm/T)\}, \tag{6.42}$$

so that,

$$a_n = 2\,\mathrm{Re}(\alpha_n),\ b_n = -2\,\mathrm{Im}(\alpha_n) \tag{6.43}$$

and when convenient, we can simply switch from one representation to the other.

Software that shifts the frequencies around has to be used carefully, as one typically rearranges the result to be physically meaningful (e.g., by placing negative frequency values in a list preceding positive frequency values with zero frequency in the center). If an inverse transform is to be implemented, one must shift back again to whatever convention the software expects. Modern software computes Fourier transforms by a so-called fast Fourier transform (FFT) algorithm, and not by the straightforward calculation of (6.30, 6.31). Various versions of the FFT exist, but they all take account of the idea that many of the operations in these coefficient calculations are done repeatedly, if the number, $N$ of data points is not prime. I leave the discussion of the FFT to the references (see also, Press et al., 1992), and will only say that one should avoid prime $N$, and that for very large values of $N$, one must be concerned about round-off errors propagating through the calculation.

*Exercise.* Consider a time series $x_m, -T/2 \leq m \leq T/2$, sampled at intervals $\Delta t$. It is desired to interpolate to intervals $\Delta t/q$, where $q$ is a positive integer greater than 1. Show (numerically) that an extremely fast method for doing so is to find $\hat{x}(s), |s| \leq 1/2\Delta t$, using an FFT, to extend the baseband with zeros to the new interval $|s| \leq q/(2\Delta t)$, and to inverse Fourier transform back into the time domain. (This is called "Fourier interpolation" and is very useful.).