2. Fourier Series and Time-Limited Functions

Suppose $x(t)$ is periodic:

$$x(t) = x(t+T) \tag{2.1}$$

Define the complex Fourier *coefficients* as

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp\left(\frac{-2\pi int}{T}\right)dt \tag{2.2}$$

Then under very general conditions, one can represent $x(t)$ in a Fourier Series:

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \exp\left(\frac{2\pi int}{T}\right). \tag{2.3}$$

Exercise. Write $x(t)$ as a Fourier cosine and sine series.

The Parseval Theorem for Fourier series is

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)^2\, dt = \sum_{n=-\infty}^{\infty} |a_n|^2, \tag{2.4}$$

and which follows immediately from the orthogonality of the complex exponentials over interval T.

Exercise. Prove the Fourier Series versions of the shift, differentiation, scaling, and time-reversal theorems.

Part of the utility of $\delta-$functions is that they permit us to do a variety of calculations which are not classically permitted. Consider for example, the Fourier transform of a periodic function, e.g., any $x(t)$ as in Eq. (2.3),

$$\hat{x}(s) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_n e^{(2\pi int/T)} e^{-2\pi ist} dt = \sum_{n=-\infty}^{\infty} \alpha_n \delta(s - n/T), \tag{2.5}$$

ignoring all convergence issues. We thus have the nice result that a periodic function has a Fourier transform; it has the property of vanishing except precisely at the usual Fourier series frequencies where its value is a $\delta-$function with amplitude equal to the complex Fourier series coefficient at that frequency.

Suppose that instead,

$$x(t) = 0, \ |t| \geq T/2 \tag{2.6}$$

that is, $x(t)$ is zero except in the finite interval $-T/2 \leq t \leq T/2$ (this is called a "time-limited" function). The following elementary statement proves to be very useful. Write $x(t)$ as a Fourier series in $|t| < T/2$, and as zero elsewhere:

$$x(t) = \begin{cases} \sum_{n=-\infty}^{\infty} \alpha_n \exp(2\pi int/T), \ |t| \leq T/2 \\ 0, \ |t| > T/2 \end{cases} \tag{2.7}$$

where

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp\left(-\frac{2\pi int}{T}\right)dt, \tag{2.8}$$

as though it were actually periodic. Thus as defined, $x(t)$ corresponds to some different, periodic function, in the interval $|t| \leq T/2$, and is zero outside. $x(t)$ is perfectly defined by the special sinusoids with frequency $s_n = n/T$, $n = 0, \pm 1, \ldots \infty$.

The function $x(t)$ isn't periodic and so its Fourier transform can be computed in the ordinary way,

$$\hat{x}(s) = \int_{-T/2}^{T/2} x(t)\, e^{-2\pi i s t} dt. \tag{2.9}$$

and then,

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(s)\, e^{2\pi i s t} ds. \tag{2.10}$$

We observe that $\hat{x}(s)$ is defined at *all* frequencies s, on the continuum from 0 to $\pm\infty$. If we look at the special frequencies $s = s_n = n/T$, corresponding to the Fourier series representation (2.7), we observe that

$$\hat{x}(s_n) = T\alpha_n = \frac{1}{1/T}\alpha_n. \tag{2.11}$$

That is, the Fourier transform at the special Fourier series frequencies, differs from the corresponding Fourier series coefficient by a constant multiplier. The second equality in (2.11) is written specifically to show that the Fourier transform value $\hat{x}(s)$, can be thought of as an amplitude density per unit frequency, with the α_n being separated by $1/T$ in frequency.

The information content of the representation of $x(t)$ in (2.7) must be the same as in (2.10), in the sense that $x(t)$ is perfectly recovered from both. But there is a striking difference in the apparent efficiency of the forms: the Fourier series requires values (a real and an imaginary part) at a countable infinity of frequencies, while the Fourier transform requires a value on the line continuum of all frequencies. One infers that the sole function of the infinite continuum of values is to insure what is given by the second line of Eq. (2.7): that the function vanishes outside $|t| \leq T/2$. Some thought suggests the idea that one ought to be able to calculate the Fourier transform at *any* frequency, from its values at the special Fourier series frequencies, and this is both true, and a very powerful tool.

Let us compute the Fourier transform of $x(t)$, using the form (2.7):

$$\hat{x}(s) = \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \alpha_n \exp\left(\frac{2\pi i n t}{T}\right) \exp(2\pi i s t)\, dt \tag{2.12}$$

and assuming we can interchange the order of integration and summation (we can),

$$\begin{aligned} \hat{x}(s) &= T \sum_{n=-\infty}^{\infty} \alpha_n \frac{\sin(\pi T(n/T - s))}{\pi T(n/T - s)} \\ &= \sum_{n=-\infty}^{\infty} \hat{x}(s_n) \frac{\sin(\pi T(n/T - s))}{\pi T(n/T - s)}, \end{aligned} \tag{2.13}$$

using Eq. (2.11). Notice that as required $\hat{x}(s) = \hat{x}(s_n) = T\alpha_n$,when $s = s_n$, but in between these values, $\hat{x}(s)$ is a weighted (interpolated) linear combination of all of the Fourier Series components.

Exercise. Prove by inverse Fourier transformation that any sum

$$\hat{x}\left(s\right)=\sum_{n=-\infty}^{\infty}\beta_n\frac{\sin\left(\pi T\left(n/T-s\right)\right)}{\pi T\left(n/T-s\right)}, \tag{2.14}$$

where β_n are arbitrary constants, corresponds to a function vanishing $t>\left|T/2\right|$, that is, a time-limited function.

The surprising import of (2.13) is that the Fourier transform of a time-limited function can be perfectly reconstructed from a knowledge of its values at the Fourier series frequencies alone. That means, in turn, that a knowledge of the countable infinity of Fourier coefficients can reconstruct the original function exactly. Putting it slightly differently, *there is no purpose in computing a Fourier transform at frequency intervals closer than* $1/T$ *where* T *is either the period, or the interval of observation.*