

Chapter 16

Boundary layer turbulence

Turbulence in the ocean and atmosphere is strongly affected by the presence of boundaries. Boundaries impose severe modifications to the momentum and buoyancy budgets. At solid boundaries, the boundary condition that the fluid velocity is zero applies to both the mean velocity and to the fluctuations. Thus the turbulent fluxes of momentum must vanish. At the ocean free surface winds apply a stress that drives strongly turbulent motions. Finally, fluxes of heat, salt, and moisture at the boundaries can generate vigorous turbulent convection. Before discussing in detail the physics of planetary boundary layers in the ocean and atmosphere, it is useful to review some fundamental results that apply to all turbulent boundary layers.

16.1 Frictional Boundary Layers

Let us consider turbulence at solid boundaries. At such boundaries, the condition that the fluid velocity is zero applies at every instant in time. Thus it applies to the mean velocity and the fluctuations separately,

$$\bar{\mathbf{u}} = 0, \quad \mathbf{u}' = 0. \quad (16.1)$$

The fact that the fluctuations drop to zero at the wall has the particular implication that the Reynolds stress vanish,

$$-\overline{u_i u_j} = 0. \quad (16.2)$$

The only stress exerted directly on the wall is the viscous one. Away from the wall, instead, turbulence generates a Reynolds stress typically large compared to the viscous stress. Tritton (chapter 5, page 337) shows in Figure 21.12 the transition between a viscous stress and a turbulent stress in a turbulent boundary layer experiment (Schubauer, *J. Appl. Physics*, 1954). The total stress parallel to the wall does not

change with distance from the wall, but there is an exchange of balance between the viscous and turbulent contributions.

Further reading: Tritton, chapter 21, 336–344

16.1.1 Turbulent motions near a wall

To simplify the algebra let us consider a parallel irrotational flow over a flat boundary. Turbulence is generated because the no-slip condition $\bar{u} = 0$ at the boundary means that a shear layer results, and vorticity is introduced into the flow. Boundary-layer flows are more complicated than free shear flows, because the importance of viscosity at the boundaries (which enforces the no-slip condition) introduces a new spatial scale in the problem. As a result there is a viscous sublayer next to the wall, whose width is set by viscous forces, and a high Re boundary layer, whose thickness is controlled by the turbulent Reynolds stresses. These two layers are separated by an inertial sublayer. The three different regions of the boundary layer are somewhat analogous to the viscous range, inertial range, and forcing ranges of isotropic, homogeneous turbulence.

1. The viscous sublayer

For distances close to the wall, i.e. $z < z_f$ where z_f is the distance at which $Re = 1$, friction is important. This can be compared to length scales $l \approx 1/k_d$ in homogeneous turbulence, where viscosity is important.

2. The inertial sublayer

At distances further away from the wall than z_f , we can neglect viscosity. Similarly, if we are not close to the edge of the boundary layer at $z = \delta$, we can assume that the flow will not depend directly on the size of the boundary layer. Therefore we have an inertial sublayer for $z_f \ll z \ll \delta$. This region is similar to the inertial range in homogeneous turbulence, where the flow is not affected by ν or by k_0 , the wavenumber of the energy input.

3. The turbulent boundary layer

The full turbulent boundary layer is determined by the maximum size of the eddies, the so-called the integral scale δ . This region corresponds to the forcing range of 3D turbulence.

4. The ambient flow

Finally at some distance $z > \delta$, the flow is no longer turbulent and we are in the irrotational ambient flow.

Further reading: Tennekes and Lumley, chapter 5, 147–163.

16.1.2 Equations of motion

We will assume a constant background flow \bar{u}_0 , which is independent of distance along the plate x and distance normal to the plate z . We assume 2-dimensional flow ($\partial/\partial y = 0$), and also assume that downstream evolution is slow. If L is a streamwise lengthscale, we are assuming $\delta/L \ll 1$, so that we can neglect variations in the streamwise direction compared to those in the vertical for averaged variables (i.e. $\partial/\partial x = 0$). Given these assumptions, the Reynolds averaged equations become,

$$\bar{w} \frac{d\bar{u}}{dz} = \frac{d}{dz} \left(\nu \frac{d\bar{u}}{dz} - \overline{w'u'} \right), \quad \frac{d\bar{w}}{dz} = 0. \quad (16.3)$$

Because of the no normal flow through the boundary, we have $\bar{w} = w' = 0$ at $z = 0$, the bottom boundary. Then from eq. (16.3b) $\bar{w} = 0$ for all z . Then eq. (16.3a) becomes,

$$\frac{d}{dz} \left(\nu \frac{d\bar{u}}{dz} - \overline{w'u'} \right) = 0. \quad (16.4)$$

Hence if we have a stress τ given by,

$$\tau = \nu \frac{d\bar{u}}{dz} - \overline{w'u'} = \left(\nu \frac{d\bar{u}}{dz} \right)_{z=0}, \quad (16.5)$$

this stress is constant throughout the boundary layer. Near the boundary the stress is dominated by the viscous term. Away from the boundary we will have,

$$\tau = -\overline{w'u'}. \quad (16.6)$$

We can define a velocity scale from this surface stress

$$u_*^2 = \tau, \quad (16.7)$$

where u_* is the **friction velocity**. Away from the boundary eq. (16.6) implies that u_* is the turbulent velocity fluctuation magnitude.

16.1.3 Viscous sublayer: law of the wall

The frictional length scale z_f is the scale at which $Re = 1$, i.e. the scale at which the viscous and turbulent stresses are of comparable magnitude. Thus the frictional length scale can be defined as,

$$z_f = \frac{\nu}{u_*}. \quad (16.8)$$

This lengthscale determines the transition between the inertial and viscous sublayers.

In the viscous sublayer $z < z_f$, the velocity must depend on z , the distance from the wall, u_* , the friction velocity and ν , the viscosity. We can write this relationship as,

$$\frac{\bar{u}}{u_*} = f\left(\frac{zu_*}{\nu}\right) \quad (16.9)$$

Note that \bar{u} has been nondimensionalized by u_* , and the distance z has been nondimensionalized by the frictional lengthscale ν/u_* . We can rewrite the relation in nondimensional form,

$$\bar{u}^+ = f(z^+) \quad (16.10)$$

where $\bar{u}^+ = \bar{u}/u_*$ and $z^+ = zu_*/\nu$.

Near a rough wall, the characteristic scale instead of being controlled by a frictional scale, it may be controlled by roughness length z_0 , if $z_0 > z_f$, and the self-similar solution in eq. (16.10) must be interpreted with $z^+ = z/z_0$.

We can also derive the exact profile of velocity in the viscous sublayer very close to the wall through the following argument. At the wall,

$$u = w = 0, \quad (16.11)$$

and taken in conjunction with the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (16.12)$$

gives,

$$\frac{\partial w}{\partial z} = 0 \text{ at } z = 0. \quad (16.13)$$

Hence,

$$\frac{d\overline{u'w'}}{dz} = \frac{d^2\overline{u'w'}}{dz^2} \text{ at } z = 0, \quad (16.14)$$

and treating the full stress as a constant in eq. (16.5),

$$\frac{d^2\bar{u}}{dz^2} = \frac{d^3\bar{u}}{dz^3} = 0 \text{ at } z = 0. \quad (16.15)$$

We thus expect there to be a significant region right next to the wall in which the velocity profile is linear,

$$\frac{\bar{u}}{u_*} = \frac{zu_*}{\nu}. \quad (16.16)$$

Tritton (chapter 5, pag 343) in Figure 21.17 shows that the linear profile is observed in experiments, next to the wall for $zu_*/\nu < 8$.

16.1.4 Turbulent boundary layer: velocity defect law

Outside the viscous sublayer, we can neglect viscosity. Thus the only dimensional parameters that enter in the problem are the turbulent velocity scale u_* , the total depth of the boundary layer δ , and the height z away from the wall. We can express this dependence as,

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\delta} g\left(\frac{z}{\delta}\right). \quad (16.17)$$

This relationship states that the mean velocity gradient, $d\bar{u}/dz$, which is the reciprocal of a transverse time scale for the mean flow, has to be of order u_*/δ and varies on spatial scales of order δ . Notice that we cannot make a similar scaling argument for the mean velocity \bar{u} and say that $\bar{u} = u_*g(z/\delta)$, because the mean velocity depends on an additional external parameter, the velocity outside the boundary layer \bar{u}_0 . We know that for $z/\delta \rightarrow \infty$, we have $\bar{u} \rightarrow \bar{u}_0$.

We can now integrate from $z = \infty$ in toward the boundary to obtain \bar{u} ,

$$\int_z^\infty \frac{d\bar{u}}{dz'} dz' = \frac{u_*}{\delta} \int_z^\infty g\left(\frac{z'}{\delta}\right) dz', \quad (16.18)$$

and hence,

$$\bar{u}(z) - \bar{u}_0 = u_* F\left(\frac{z}{\delta}\right), \quad (16.19)$$

or in nondimensional form,

$$\bar{u}^+ - \bar{u}_0^+ = F(\zeta), \quad (16.20)$$

where $\zeta = z/\delta$. This is a similarity solution for \bar{u}^+ , which assumes that as the boundary layer changes size, or for different boundary layers \bar{u}^+ has the same form. This similarity solution is only valid outside of the viscous boundary layer, and cannot satisfy the boundary condition $\bar{u} = 0$ at the wall.

16.1.5 Inertial sublayer: logarithmic layer

Thus far we have two different laws for \bar{u}^+ . One applies close to the wall in the viscous sublayer and satisfies the no-slip condition $\bar{u} = 0$. The other applies further away from the wall and is not guaranteed to satisfy the no-slip boundary condition at the wall; actually it turns out that away from the wall $u_* \ll \bar{u}_0$ and thus $\bar{u} - \bar{u}_0 \approx \bar{u}_0$. This indicates that the viscous sublayer with very steep gradients is required in order to satisfy the boundary conditions. Of course the velocity doesn't suddenly jump from one scaling behavior to another - there is a transition region. In this transition region we expect both the law of the wall and the velocity defect law to apply.

From eq. (16.10) we expect that,

$$\frac{d\bar{u}^+}{dz^+} = \frac{df}{dz^+}. \quad (16.21)$$

From eq. (16.20) instead we have,

$$\frac{d\bar{u}^+}{dz^+} = \frac{d\zeta}{dz^+} \frac{dF}{d\zeta} = \frac{\zeta}{z^+} \frac{dF}{d\zeta}, \quad (16.22)$$

where we used the fact that $\zeta = z/\delta$ and $z^+ = zu_*/\nu$. In this overlap region these two expressions must be equal so,

$$\frac{df}{dz^+} = \frac{\zeta}{z^+} \frac{dF}{d\zeta}, \quad (16.23)$$

and rearranging terms,

$$z^+ \frac{df}{dz^+} = \zeta \frac{dF}{d\zeta}. \quad (16.24)$$

The right hand side of eq. (16.24) depends only on ζ and the left hand side can depend only on z^+ . This can only be true only if both sides are equal to a constant,

$$z^+ \frac{df}{dz^+} = \zeta \frac{dF}{d\zeta} = \frac{1}{\kappa}, \quad (16.25)$$

where κ is the **Von Karman constant**. This implies that

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\kappa z} \quad (16.26)$$

so that in this region the only important quantities are u_* and z . Then in this transition region, the inertial sublayer, the flow is unaware both of viscosity and of the size of the boundary layer δ - just as in the inertial range isotropic homogeneous 3D turbulence is unaware of viscosity or of the integral scale of the forcing.

Integrating eq.(16.25) we have,

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \log\left(\frac{u_* z}{\nu}\right) + C_1, \quad (16.27)$$

and ,

$$\frac{\bar{u} - \bar{u}_0}{u_*} = \frac{1}{\kappa} \log\left(\frac{z}{\delta}\right) + C_2. \quad (16.28)$$

The region where this applies ($\zeta \ll 1$, $z^+ \gg 1$) is known as the logarithmic layer.

Near a rough boundary, the equivalent of 16.27 would be,

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \log\left(\frac{z}{z_0}\right) + C_1, \quad (16.29)$$

with z_0 , the roughness length, taking the place of $z_f = \nu/u^*$, the frictional lengthscale.

Hinze (chapter 7, pag. 477) in figure 1.6.1 show the mean velocity distribution adjacent to a smooth wall, showing the logarithmic distribution away from the viscous region next to the wall and the linear region in the viscous sublayer from a composite of different laboratory experiments.

The value of the Von Karman constant has been measured in a variety of laboratory flows that indicate a universal value of 0.41. Some early measurements in the atmosphere (Businger et al., 1971) suggested a much smaller value of 0.35, and this led to speculations for a while that the constant might not be universal, but instead a function of salient nondimensional numbers in the flow (for example the Rossby number). Careful reexamination of the errors involved (Hogstrom, 1996) and more recent observations (Zhang, 1988) indicate that the constant is indeed a constant with a value around 0.40 ± 0.01 .

16.1.6 Coherent structures

From lecture 1 we emphasized the fact that any turbulent flow involves large eddies with a coherent structure: turbulent boundary layers are no exceptions to the rule. Tritton (chapter 21, pages 344–350) reviews the observations and properties of coherent structures in boundary layers.

16.2 Stratified Boundary Layers

At a boundary, in addition to surface stresses acting as a source of vorticity, we may also have buoyancy forcing (for example atmospheric heating and cooling at the ocean surface, or radiant heating at the land surface). Whether the dynamics of the turbulent boundary layer are mostly affected by buoyancy forcing or by shear effects can be quantified in terms of the flux Richardson number,

$$R_f = \frac{\overline{w'b'}}{\overline{u'w'\partial\bar{u}/\partial z}}. \quad (16.30)$$

We have seen in the lecture on energetics that the flux Richardson number represents the ratio between TKE buoyancy production and TKE shear production. Recall that positive values of R_f imply stable stratification, when TKE is lost to PE, and negative values imply unstable stratification, when PE generates TKE. Hence if $R_f > 1$, we expect the surface buoyancy fluxes to suppress the boundary layer turbulence, while if $R_f < -1$ we expect the boundary turbulence to be dominated by convective mixing,

rather than shear generated turbulence. For $-1 < R_f < 1$, the shear production of turbulence dominates and frictional boundary layer theory applies.

We can express the transition between buoyancy generated turbulence to shear generated turbulence also in terms of vertical length scales. Using boundary layer scaling we have,

$$\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{\kappa z}, \quad \overline{w'w'} = -u_*^2. \quad (16.31)$$

The flux Richardson number is then,

$$R_f = -\frac{\kappa \overline{w'b'}}{u_*^3} z. \quad (16.32)$$

When $|R_f| = 1$, then the buoyancy production/loss of TKE is of equal magnitude to shear production. This occurs at a lengthscale,

$$L_b = \frac{u_*^3}{\kappa |\overline{w'b'}|}. \quad (16.33)$$

If the buoyancy flux is supplied through a surface flux, then the minimum value of L_b is,

$$L_b = \frac{u_*^3}{\kappa |\overline{w'b'_0}|}. \quad (16.34)$$

This is the **Monin-Obukhov lengthscale**. If $\overline{w'b'_0} > 0$ the flux is destabilizing. Then for distances from the boundary $z < L_b$, the shear production dominates, while for distances $z > L_b$, buoyant convection dominates. If $\overline{w'b'_0} < 0$ for distances $z > L_b$ the turbulence is damped by the stable stratification.

16.2.1 The velocity profile

In a turbulent boundary layer forced with buoyancy fluxes, velocity gradients above the viscous sublayer, depend on $\overline{w'b'_0}$, represented by L_b , as well as on u_* and z . Dimensional analysis leads to the following expanded version of the logarithmic profile,

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\kappa z} \phi\left(\frac{z}{L_b}\right), \quad (16.35)$$

where $\phi(z/L_b)$ is an unspecified function. Under neutral condition, when stratification is neither stable or unstable, i.e. at vanishing $\overline{w'b'_0}$, hence $z/L_b \rightarrow 0$, and $\phi(z/L_b)$ must tend to unity. Large positive $\overline{w'b'_0}$ generates vigorous convection and reduce stress-induced mechanical turbulence to insignificance. At moderate positive $\overline{w'b'_0}$, or z/L_b of order -1 , mechanical and convective turbulence are both important and eq. (16.35) is useful. At the other extreme, large negative buoyancy flux overwhelms

mechanical turbulence to the point of completely suppressing it. At moderately high negative $\overline{w'b'}_0$ (i.e. positive and small z/L_b), eq. (16.35) is still valid. The negative buoyancy flux in the TKE equation implies that work is expended into against gravity to raise heavier fluid up from lower levels. The PE production must balance the loss of TKE, resulting in less vigorous shear turbulence, and sharper mean velocity gradients.

Boundary layer meteorologists have explored buoyancy effects on the atmospheric surface layer and proposed several different empirical formulae for the function $\phi(z/L_b)$, separately for stable and unstable conditions. A few are reported in Csanady (Air-sea interaction, chapter 1.4.4).

Further reading: Tennekes and Lumley, chapters 2.5, 3.4 and 5; Lesieur, chapter 4, section 1.2.6; Hinze chapter 7; Phillips, chapter 6.6.

16.2.2 The buoyancy profile

We have seen that in turbulent flows the Reynolds fluxes of tracers like buoyancy $\overline{w'b'}$ are the mean vehicles of transport to or from the boundaries, just as the Reynolds stress is for momentum. Much alike in the case of momentum, the final step at the interface has to be transfer by molecular diffusion. This means that diffusive boundary layers develop at the boundary. Turbulent eddy motions confine these boundary layers to the immediate vicinity of the interface, counteracting the tendency of the diffusive boundary layers to grow.

Well above the diffusive boundary layers, the influence of molecular properties becomes imperceptible and we have an inertial sublayer. Under the same assumptions considered for the momentum budget, gradients of mean buoyancy then depend only on the buoyancy flux and the two scales of turbulence,

$$\frac{d\bar{b}}{dz} = \text{func}(\overline{w'b'}, u_*, z). \quad (16.36)$$

There are four variables in this equation, and three units of length, time, and buoyancy. Hence, they can be combined into a single nondimensional variable that should be constant. We introduce a buoyancy scale as $b_* = -\overline{w'b'}/u_*$, the negative sign being chosen so that b_* has the same sign as $b(z) - b_s$, where b_s is the buoyancy at the solid boundary. Eq. (16.36) in nondimensional form is then,

$$\frac{d\bar{b}}{dz} = \text{const} \frac{b_*}{z}, \quad (16.37)$$

which integrates to a logarithmic law,

$$\frac{\bar{b} - b_s}{b_*} = \frac{1}{\kappa} \log\left(\frac{z}{\delta}\right), \quad (16.38)$$

where δ is the turbulent boundary layer thickness and κ is the Von Karman constant.

The above scalings hold as long as the buoyancy fluxes are small to affect turbulent transport. The correction to eq. (16.39) for situations where buoyancy fluxes are not negligible take the form,

$$\frac{d\bar{b}}{dz} = \frac{b_*}{\kappa z} \phi_b \left(\frac{z}{L_b} \right), \quad (16.39)$$

with ϕ_b a function to be determined from observations.

16.3 Planetary Boundary Layers

The boundary layers in geophysical flows are also affected by rotation through the Coriolis force. This is discussed by Tennekes and Lumley, chapter 5.3.