2-D Turbulence

One might think the two-dimensional problem might be a simpler, more tractable version of 3-D turbulence. However, the dynamics turn out to be quite different. To see why this is the case, let's consider the vorticity dynamics of each kind of flow. From the Navier-Stokes equations for a homogeneous fluid

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u} + \boldsymbol{\zeta} \times \mathbf{u} &= -\nabla(p + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - \nu\nabla\times\boldsymbol{\zeta} \\ \nabla\cdot\mathbf{u} &= 0 \\ \boldsymbol{\zeta} &= \nabla\times\mathbf{u} \end{aligned} \tag{1}$$

we can derive the vorticity equation

$$\frac{\partial}{\partial t}\boldsymbol{\zeta} + \nabla\times(\boldsymbol{\zeta}\times\mathbf{u}) = \nu\nabla^2\boldsymbol{\zeta} \tag{2}$$

and the energy equation

$$\frac{\partial}{\partial t}E + \nabla\cdot[\mathbf{u}(p+E)] = -\nu\mathbf{u}\cdot\nabla\times\boldsymbol{\zeta}$$

or

$$\frac{\partial}{\partial t}E + \nabla\cdot[\mathbf{u}(p+E) + \nu\boldsymbol{\zeta}\times\mathbf{u}] = -2\nu Z \tag{3}$$

with $E = \frac{1}{2}\mathbf{u}\cdot\mathbf{u}$ being the energy and $Z = \frac{1}{2}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}$ being the enstrophy. Turbulence in the presence of boundaries can flux energy through the walls, but for homogeneous turbulence the flux term must on the average vanish and the dissipation is just $2\nu Z$. **Dissipation and energy cascades are closely tied to the rotational nature of a turbulent flow.**

In the classic picture, we force the energy at a certain rate, which is then balanced off by dissipation. But the spectrum depends only on the forcing rate and k, not upon viscosity. For this to hold, the enstrophy must grow as the viscosity decreases. This is consistent with the Kolmogorov spectrum: if $E(k) \sim k^{-5/3}$, $Z(k) \sim k^2E(k) \sim k^{1/3}$. As the viscosity decreases and we excite smaller and smaller scales, the enstrophy will become larger and larger to compensate.

The enstrophy equation indeed has suitable source terms:

$$\frac{\partial}{\partial t}Z + \nabla\cdot(\mathbf{u}Z - \nu\nabla Z) = \zeta_i S_{ij}\zeta_j - \nu(\nabla_i\zeta_j)^2 \tag{4}$$

where $S_{ij} = \frac{1}{2}[\nabla_i u_j + \nabla_j u_i]$ is the rate of strain tensor. Therefore, if the vorticity vector is on average aligned with the directions where the strain is causing extension rather than contraction, the enstrophy will increase. In a 3-D turbulent flow, the vorticity is indeed on average undergoing stretching since this term ends up balancing the sign-definite dissipation terms.

Two-D

The two-dimensional system differs significantly in its vorticity dynamics. Since $\mathbf{u}\cdot\hat{\mathbf{z}} = 0$ and $\frac{\partial}{\partial z}\mathbf{u} = 0$, the vorticity is purely vertical $\boldsymbol{\zeta} = \zeta\hat{\mathbf{z}}$. The enstrophy production term becomes

$$\zeta_i S_{ij}\zeta_j = \zeta^2 S_{33} = \zeta^2 \frac{\partial w}{\partial z} = 0$$

In a 2D system, the energy and enstrophy will decay monotonically

$$\frac{\partial}{\partial t}E = -2\nu Z \quad , \qquad \frac{\partial}{\partial t}Z = -\nu|\nabla\zeta|^2 \tag{5}$$

Since the enstrophy is bounded by its initial value, the energy dissipation rate will decrease in proportion to the viscosity; an energy cascade in the Kolmogorov sense will not occur.

However the term giving dissipation of enstrophy will have a non-trivial generation term; this follows from the vorticity equation

$$\frac{\partial}{\partial t}\zeta + \mathbf{u}\cdot\nabla\zeta = \nu\nabla^2\zeta$$

and examination of the dissipation term $-\nu|\nabla\zeta|^2$ If we let $\mathbf{g} = \nabla\zeta$ and $G = \frac{1}{2}|\nabla\zeta|^2$ then

$$\frac{\partial}{\partial t}\mathbf{g} + \nabla[\mathbf{u}\cdot\nabla\zeta] = \nu\nabla[\nabla^2\zeta]$$

and

$$\frac{\partial}{\partial t}G + \nabla\cdot[\mathbf{u}G - \nu\nabla G] = -g_i S_{ij} g_j - \nu(\nabla_i g_j)^2 \tag{6}$$

If the contours of vorticity are aligned with the axis of extension, the gradients will be aligned with the axis of compression and the vorticity lines will be pushed closer together, increasing the mean square gradient. Thus enstrophy can cascade, since there is a source term for vorticity gradients.

Spectrum – dimensional argument

These results suggest that we could derive a spectrum from an enstrophy cascade argument: $E(k)$ (dimensions L^3/T^2) should depend on k (L^{-1}) and the rate of enstrophy dissipation $\eta = \nu(\nabla_i g_j)^2$ (which has dimensions T^{-3}). The result is

$$E(k) = const \times k^{-3}\eta^{2/3}$$

with the enstrophy spectrum also decreasing at small scales $Z(k) \sim k^{-1}$.

2D simulations

What does 2-D turbulence look like? We show results from a 512×512 2-D pseudospectral code solving the vorticity equation and the inversion of vorticity to find the streamfunction.

$$\frac{\partial}{\partial t}\zeta + J(\psi,\zeta) = filter$$

$$\zeta = \nabla^2\psi$$

`Demos, Page 2:` 2D <*psi*> <*zeta*> `Demos, Page 2:` Statistics <*spectra*> <*pdf of zeta*> `Demos, Page 2:` Averages <*E and Z*> <*variance and kurtosis*> The streamfunction shows a marked increase in scale – the so-called "inverse cascade" while the vorticity collects into strong isolated vortices with filamentary structure in between. The PDF and the kurtosis of the vorticity field shows the non-Gaussian nature of the flow quite clearly.

Inverse cascade

We can quantify the inverse cascade using an argument of Peter Rhines: consider the average scale, defined as

$$\overline{k} \equiv \frac{\int_0^\infty kE(k)}{\int_0^\infty E(k)}$$

We note that $E = \int E(k)$ and $Z = \int k^2 E(k)$ are conserved in inviscid motion. If we presume the energy is initially near k_0 and is spreading then

$$\frac{\partial}{\partial t}\frac{1}{E}\int (k-k_0)^2 E(k) > 0$$

Therefore

$$\frac{\partial}{\partial t}\left[\int k^2 E(k) - 2k_0 \int kE(k) + k_0^2 E\right] > 0$$

$$\frac{\partial}{\partial t}[Z - 2k_0\overline{k}E + k_0^2 E] > 0$$

$$-2k_0 E\frac{\partial}{\partial t}\overline{k} > 0$$

or

$$\frac{\partial}{\partial t}\overline{k} < 0$$

As the energy spreads in the spectrum its mean scale increases. `Demos, Page 3:` mean k <*kbar vs time*>

Multiple power laws

We now have two possible power laws: the K41 law

$$E \sim \epsilon^{2/3} k^{-5/3}$$

and the enstrophy cascade law

$$E \sim \eta^{2/3} k^{-3}$$

In the first case, the enstrophy transfer rate must be zero since then $\eta \sim k^2 \epsilon$ which will not be independent of k unless the coefficient is zero. Similarly, the energy cascade rate will be zero in the second case.

Calculations of the transfer rates (c.f. Kraichnan, 1967) show that K41 gives upscale energy transfer while k^{-3} gives downscale enstrophy transfer. We might consider using the former in the range $k < k_{injection}$ and the latter for smaller scales $k > k_{injection}$.

Problems with the power law spectra

The assumption in the similarity models is that the turn-over time at scale k depends on the net shear/ strain at that scale

$$s^2 = \int_{k_0}^{k} Z(k) dk$$

For K41, this gives

$$s^2 \sim k^{4/3} - k_0^{4/3}$$

which is dominated by the contribution near k — the transfers are local. The estimate of strain converges as $k_0 \to 0$ and 60% is generated from $k/2$ to k. For the k^{-3} case, however,

$$s^2 \sim \log \frac{k}{k_0}$$

which diverges; indeed, the contribution from $k/8$ to $k/4$ is the same as from $k/4$ to $k/2$ and as that from $k/2$ to k. Thus the transfers are **not** local and the argument is not consistent.

Vortex dynamics

Onsager (19xx) realized that vortex dynamics might indeed be a significant part of 2-D turbulence. For the inviscid problem, a point vortex

$$\zeta = s\delta(\mathbf{x} - \mathbf{x}_0(t)) \quad , \quad \psi = sG(\mathbf{x} - \mathbf{x}_0(t))$$

is a basic solution in the sense that the inversion formula from vorticity to streamfunction involves the Green's function $G(\mathbf{x}) = \log(|\mathbf{x}|)/2\pi$

$$\psi = \int G(\mathbf{x} - \mathbf{x}')\zeta(\mathbf{x}')d^3\mathbf{x}'$$

To derive the formula for G, we locate the origin at the vortex, use symmetry to replace $\nabla^2\psi$ with $\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\psi$ and use the free solution away from $r = 0$. Integrating over a small disk centered at $r = 0$ gives the constant in front.

A single vortex does not advect itself: we can think of the motion of the point as associated with the average velocity in a small disk centered at $\mathbf{x}_0$. This flow is produced by all the other vortices in the flow. Thus, if we have a set of vortices at positions $\mathbf{x}_i$, we have

$$\frac{d}{dt}\mathbf{x}_i = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) \sum_{j \neq i} s_j G(\mathbf{x}_i - \mathbf{x}_j)$$

Kirchhoff (1876) realized that this could be written as a Hamiltonian system

$$H = -\frac{1}{2} \sum_{i \neq j} \sum s_i G(\mathbf{x}_i - \mathbf{x}_j) s_j$$

$$\frac{\partial}{\partial t} F = \{F, H\}$$

$$\{F, G\} = \sum \frac{1}{s_i} \left(\frac{\partial F}{\partial x_i} \frac{\partial G}{\partial y_i} - \frac{\partial F}{\partial y_i} \frac{\partial G}{\partial x_i} \right)$$

For a small number of vortices, this system is integrable: it has 4 conserved properties (energy, x and y centers of mass, angular momentum) vs. 2 unknowns per vortex. With more than three vortices, however, the motions can become chaotic.

`Demos, Page 5: Point vortex simulations` *<n=3>* *<delta solutions>* *<n=4>* *<delta>* *<single vortex trajectory>* *<long example>*

Onsager(1941) considered the statistical properties of a many vortex system and predicted clumping of like-signed vortices would result, corresponding to a cascade to larger scales.

Vortex dynamics

To understand the merger process, let us consider the inviscid dynamics of a single vortex embedded in a shear or strain field. We can represent the vorticity as

$$\zeta = \begin{cases} q_0 + \Delta & \mathbf{x} \in D \\ q_0 & \text{else} \end{cases}$$

and the flow as

$$\psi = \frac{1}{2} q_0 y^2 + \psi'$$

with

$$\psi'(\mathbf{x}) = \Delta \iint_D d^2\mathbf{x}' G(\mathbf{x} - \mathbf{x}')$$

The points on the boundary of D are material points since they separate fluid with different vorticities. Furthermore, the velocities can be written as a line integral around the patch boundary

$$u = -\frac{\partial}{\partial y}\psi = -\Delta \iint_D \frac{\partial}{\partial y} G(\mathbf{x} - \mathbf{x}') = \Delta \iint_D \frac{\partial}{\partial y'} G(\mathbf{x} - \mathbf{x}')$$

$$u = \oint_{\partial D} G(\mathbf{x} - \mathbf{x}')\hat{\mathbf{y}}' \cdot \hat{\mathbf{n}}' = -\oint_{\partial D} G(\mathbf{x} - \mathbf{x}')\hat{\mathbf{x}}' \cdot \hat{\mathbf{t}}'$$

$$v = -\oint_{\partial D} G(\mathbf{x} - \mathbf{x}')\hat{\mathbf{x}}' \cdot \hat{\mathbf{n}}' = -\oint_{\partial D} G(\mathbf{x} - \mathbf{x}')\hat{\mathbf{y}}' \cdot \hat{\mathbf{t}}'$$

Thus the evolution of the vorticity contours can be posed as a dynamical system (in fact a Hamiltonian one). We'll analyze linearized versions but show fully nonlinear examples.

Perturbed circular vortex

We shall consider a vortex patch

$$\zeta = q_0 + \Delta \mathcal{H}(a + \eta(\theta, t) - r)$$

(where $\mathcal{H}$ is the step function) and define the circular state

$$\nabla^2 \overline{\psi} = q_0 + \Delta \mathcal{H}(a - r) \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\overline{v} - \frac{1}{r^2}\overline{v} = -\Delta \delta(a - r)$$

with $\psi = \frac{1}{2} q_0 y^2 + \overline{\psi} + \psi'$. The perturbation streamfunction satisfies

$$\nabla^2 \psi' = \Delta \left[\mathcal{H}(a + \eta - r) - \mathcal{H}(a - r)\right] \simeq \Delta \eta \delta(a - r)$$

and the condition that the edge be a material surface is

$$\frac{\partial}{\partial t}\eta = -\frac{1}{a + \eta}\frac{d}{d\theta}\psi(a + \eta(\theta, t), \theta, t)$$

$$\simeq -\frac{1}{a}\frac{d}{d\theta}\left[-\frac{q_0 a^2}{4}\cos 2\theta + \overline{v}(a)\eta(\theta, t) + \psi'(a, \theta, t)\right]$$

Free modes: In terms of the Green's function

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}g_n(r,a) - \frac{n^2}{r^2}g_n(r,a) = \frac{1}{a}\delta(a-r)$$

we have

$$\overline{v} = -\Delta\, a\, g_1(r,a) \quad , \quad \psi'_n = \Delta\, a\, \eta_n\, g_n(r,a)$$

where η_n is the n^{th} Fourier component in angle θ. The kinematic equation then gives

$$\frac{\partial}{\partial t}\eta_n = \Delta\left[g_1(a,a) - g_n(a,a)\right]\frac{\partial}{\partial \theta}\eta$$

This has solutions $\eta_n \sim \exp(\imath n(\theta - \Omega t))$ with

$$\Omega = -\Delta\left[g_1(a,a) - g_n(a,a)\right]$$

Using

$$g_n(r,a) = -\frac{1}{2n}\begin{cases}(r/a)^n & r < a\\ (a/r)^n & r > a\end{cases}$$

gives high modes being advected and low modes moving more slowly propagating against the flow because of the vorticity gradient

$$\Omega = \frac{\Delta}{2}\left[1 - \frac{1}{n}\right]$$

`Demos, Page 7: Vortex waves` *<circular>* *<go>* *<n=2>* *<go>* *<n=3>* *<go>* *<n=4>* *<go>* *<n=8>* *<go>*

Shear flow: A background shear introduces a forcing of the $n = 2$ mode. If we therefore limit ourselves to this mode, we have

$$\frac{\partial}{\partial t}\eta_2 = -\frac{q_0 a}{2}\sin 2\theta - \Omega_2\frac{\partial}{\partial \theta}\eta_2$$

This has a steady solution with the vortex elongated along the shear when the sense of rotation of the shear is the same as the vortex or perpendicular when they have opposite signs

$$\eta_2 = \frac{q_0 a}{4\Omega}\cos 2\theta = \frac{q_0 a n}{2\Delta(n-1)}\cos 2\theta$$

In general η_2 will oscillate around this value, depending on the initial condition. Kida (1981) demonstrated that an exact nonlinear solution to the problem is elliptical with the aspect ratio and orientation changing with time. `Demos, Page 7: Vortices in shear` *<elliptical>* *<go>* *<strong shear>* *<go>* *<weak shear>* *<go>* *<balanced>* *<go>* *<adverse>* *<go>* *<adverse weak>* *<go>*

Two vortices:

If two vortices are separated widely enough, we can ignore all but the monopole field of the second vortex in the vicinity of the first. In addition, we look at the first vortex in a co-rotating frame. In that case the background streamfunction field looks like

$$\psi_0 = \frac{\Delta a^2}{4}\ln[x^2+(y+R)^2] - \frac{\Delta a^2}{2R^2}[x^2+(y+\frac{R}{2})^2]$$
$$= \frac{\Delta a^2}{4}\ln[r^2+2Rr\sin\theta+R^2] - \frac{\Delta a^2}{2R^2}[r^2+rR\sin\theta+\frac{R^2}{4}]$$

The first term in the Taylor series assuming $R >> a$ is

$$\psi_0 \sim -\frac{\Delta a^4}{2R^2}\sin^2\theta = \frac{\Delta a^4}{4R^2}\cos 2\theta$$

Thus the vortex is embedded in an opposing shear and can be elongated until it meets the other vortex. In linear theory, the boundary displacement is $\Delta a^3/4R^2\Omega_2$ and crosses the centerline when $\eta = R/2 - a$; this gives a critical separation of 2.4 radii. In fact, merger occurs for separations less than 3.3 radii. `Demos, Page 8: Vortices in neighbor field` *<R=4>* *<go>* *<R=3>* *<go>* *<R=3.5>* *<go>* *<R=4 full>* *<R=3.5 full>* *<R=3.45 full>* *<R=3.4 full>* *<R=3.3 full>*