Chapter 12

Vorticity and quasi-geostrophy

Supplemental reading:

Holton (1979), chapters 4, 6

Houghton (1977), sections 8.4–6

Pedlosky (1979), chapter 2, sections 3.10, 3.12, 3.13

12.1 Preliminary remarks

In the preceding chapter we saw that β plays a major role in large-scale motions of the atmosphere. We also referred to β as the gradient of that contribution to a fluid's *vorticity* due to the Earth's rotation. We will now briefly consider what exactly is vorticity.

Recall that in particle mechanics, conservation of energy and momentum both play important roles. In a fluid, however, momentum is not in general conserved because of the presence of pressure forces. To be sure, in symmetric circulations, zonal angular momentum is conserved (in the absence of friction), but then $\frac{\partial p'}{\partial x} = 0$ by definition. The question we will consider is whether there is anything a fluid conserves instead of momentum.

12.1.1 Interpretation of vorticity

A clue is obtained from the following 'quasi-fluid' equation:

$$\frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho}\nabla p, \tag{12.1}$$

where $\rho = \text{const}$. Taking the curl of (12.1) eliminates ∇p and leaves us with

$$\frac{\partial}{\partial t}(\nabla \times \vec{u}) = 0. \tag{12.2}$$

In this 'quasi-fluid' $\nabla \times \vec{u}$ could be considered as conserved. $\nabla \times \vec{u}$ is called vorticity. Vorticity can be interpreted as twice the instantaneous local rotation rate of an element of fluid. This is easily seen in two-dimensional flow. With reference to Figure 12.1,

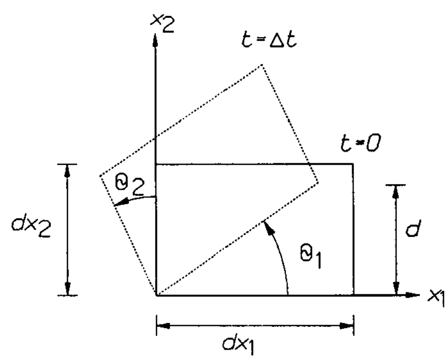

Figure 12.1: A rectangular element of fluid at $t = 0$ is deformed and rotated by the fluid flow into the rhomboidal element at $t = dt$.

$$d = \frac{\partial u_2}{\partial x_1} dx_1 dt$$

$$\Theta_1 = \frac{d}{dx_1} = \frac{\partial u_2}{\partial x_1} dt.$$

Similarly,

$$\Theta_2 = -\frac{\partial u_1}{\partial x_2} dt.$$

Now $\frac{d}{dt}(\Theta_1 + \Theta_2) \equiv 2\frac{d\alpha}{dt} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$, which is what we set out to show.

The real equations of motion are more complicated than (12.1), but as we shall see a quantity closely related to vorticity is, in fact, conserved in inviscid, adiabatic fluids.

12.2 Vorticity in the shallow water equations

Let us first consider the shallow water equations introduced in Chapter 4. This time, however, we will consider the nonlinear shallow water equations. We will retain the β-plane geometry.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -gZ_x \tag{12.3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -gZ_y \tag{12.4}$$

$$Z\nabla \cdot \vec{u} + \frac{DZ}{Dt} = 0. \tag{12.5}$$

To eliminate Z_x and Z_y in (12.3) and (12.4) (Z is the counterpart of pressure), we differentiate (12.3) with respect to y, and (12.4) with respect to x, and subtract the resulting equations:

$$\begin{aligned}\frac{\partial^2 u}{\partial t \partial y} \quad &+ \quad \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \\ &+ \quad v\frac{\partial^2 u}{\partial y^2} - f\frac{\partial v}{\partial y} - \beta v = -g\frac{\partial^2 Z}{\partial x \partial y}\end{aligned} \tag{12.6}$$

$$\begin{aligned}\frac{\partial^2 v}{\partial t \partial x} \quad &+ \quad \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \\ &+ \quad v\frac{\partial^2 v}{\partial x \partial y} + f\frac{\partial u}{\partial x} = -g\frac{\partial^2 Z}{\partial x \partial y}\end{aligned} \tag{12.7}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$+\beta v+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$+\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0 \tag{12.8}$$

or

$$\frac{D\zeta}{Dt}+\beta v+f\nabla\cdot\vec{u}+\zeta\nabla\cdot\vec{u}=0, \tag{12.9}$$

where ζ = vertical component of vorticity = $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$.
Equation 12.9 may be rewritten

$$\frac{D}{Dt}(\zeta+f)+(\zeta+f)\nabla\cdot\vec{u}=0. \tag{12.10}$$

The quantity $\zeta+f$ is called *absolute vorticity* while ζ is called *relative vorticity.* (Why ?) Using (12.5), (12.10) becomes

$$\frac{D}{Dt}(\zeta_a)-\frac{\zeta_a}{Z}\frac{DZ}{Dt}=0$$

or

$$\frac{D}{Dt}\left(\frac{\zeta_a}{Z}\right)=0, \tag{12.11}$$

where $\zeta_a=\zeta+f$. The quantity ζ_a/Z, known as *absolute potential vorticity,* is conserved by the shallow water equations. The existence of Rossby waves is closely related to the conservation of vorticity or potential vorticity. Recall that the shallow water equations allow both gravity waves and Rossby waves. If, however, we put a rigid lid on the fluid we will eliminate gravity waves. In such a situation,

$$\frac{DZ}{Dt}=0$$

and (12.11) becomes

$$\frac{D\zeta_a}{Dt} = \frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) = 0. \tag{12.12}$$

Equation 12.5 becomes

$$\nabla \cdot \vec{u} = 0 \tag{12.13}$$

which implies the existence of a stream function such that

$$u = -\psi_y \tag{12.14}$$

$$v = \psi_x. \tag{12.15}$$

12.2.1 Filtered Rossby waves

If we assume a constant basic flow, u_o, and linearizable perturbations on this flow, Equation 12.12 becomes

$$\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)(\nabla^2\psi + f) + \psi_x\beta = 0. \tag{12.16}$$

If we assume further that the perturbations are of the form

$$\sin \ell y \; e^{ik(x-ct)}$$

then (12.16) becomes

$$ik(u_0 - c)(-k^2 - \ell^2)\psi + ik\psi\beta = 0$$

or

$$c = u_0 - \frac{\beta}{k^2 + \ell^2}\,, \tag{12.17}$$

which is simply the equation for non-divergent Rossby waves. The mechanism of such waves is shown in Figure 12.2. Recall that ζ_a consists in both relative vorticity and f; f increases with y. Now if $\zeta = 0$ at $y = y_0$ and an element is displaced to a positive y, a negative ζ (clockwise rotation) will be induced

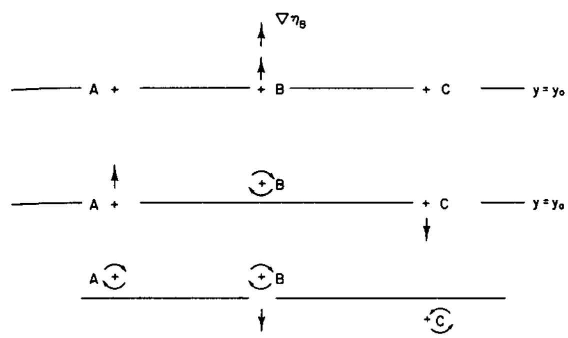

Figure 12.2: The position of the three-point vortices A, B and C at three successive times. Initially colinear and positioned along an isobar, B is displaced upwards, producing velocities at A and C which move them as shown. The vorticity induced on A and C produces a velocity at B tending to restore it to its original position. After Pedlosky (1979).

to counteract the increasing f. The result will be a disturbance whose phase propagates westward relative to u_0.

We next consider what happens if we restore a free surface. If we linearize (12.11) about a constant u_0 basic state we get

$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right)\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) + v'\beta - \frac{f}{Z_0}\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right) Z' = 0. \qquad (12.18)$$

Now the linearization of Equations 12.3 and 12.4 could be used to relate u' and v' to Z', but the resulting dispersion relation would be cubic in c. However, from the exercises we know that u' and v' in a Rossby wave are approximately geostrophic; that is,

$$v' \cong \frac{g}{f} Z'_x \qquad (12.19)$$

and

$$u' \cong -\frac{g}{f} Z'_y. \qquad (12.20)$$

Let's see what happens if we substitute (12.19) and (12.20) into (12.18):

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\left(\frac{g}{f}(Z'_{xx}+Z'_{yy})\right)+\frac{g}{f}\beta Z'_x-\frac{f}{Z_0}\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)Z'=0. \quad (12.21)$$

Again, assuming solutions of the form

$$\sin \ell y \; e^{ik(x-ct)},$$

(12.21) becomes

$$-(u_0-c)\frac{g}{f}(k^2+\ell^2)+\frac{g\beta}{f}-\frac{f}{Z_0}(u_0-c)=0$$

or

$$-(u_0-c)(k^2+\ell^2)+\beta-\frac{f^2}{gZ_0}(u_0-c)=0$$

or

$$c=u_0-\frac{\beta}{k^2+\ell^2+\frac{f^2}{gZ_0}}, \quad (12.22)$$

which is precisely the dispersion relation for divergent Rossby waves[1]. We seem to have found a way of exploiting geostrophy to suppress gravity waves while retaining the time evolution associated with Rossby waves. Our next step is to make this procedure systematic.

12.3 Quasi-geostrophic shallow water theory

Once one knows what one is after, scaling affords a convenient way to make things systematic. It will become transparently clear that if one does not know what one wants *a priori*, scaling is not nearly so effective!

[1]Note that in the non-divergent case, we automatically have a streamfunction, so that there is no need for a quasi-geostrophic approximation in order to obtain (12.17). Equivalently, the non-divergent case does not have surface gravity waves to filter out.

Let us go through the ritual of scaling the dependent and independent variables in Equations 12.3–12.5 as follows:

$$\begin{aligned}
u &= Uu' \\
v &= Uv' \\
x &= Lx' \\
y &= Ly' \\
Z &\equiv \underbrace{\bar{Z}}_{mean\ depth} + \tilde{Z}, \text{ and } \tilde{Z} = HZ' \\
f &= f + \beta y = f + \beta L y' \\
t &= Tt'.
\end{aligned}$$

In terms of dimensionless variables (12.3)–(12.5) become

$$\begin{aligned}
\frac{U}{T}\frac{\partial u'}{\partial t'} &+ \frac{U^2}{L}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) \\
&- fU\left(1 + \frac{\beta L}{f}y'\right)v' = -g\frac{H}{L}Z'_{x'}
\end{aligned}$$

or

$$\frac{1}{fT}\frac{\partial u'}{\partial t'} + \frac{U}{fL}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) - \left(1 + \frac{\beta L}{f}y'\right)v' = -\frac{gH}{fUL}Z'. \qquad (12.23)$$

Similarly,

$$\begin{aligned}
\frac{1}{fT}\frac{\partial v'}{\partial t} &+ \frac{U}{fL}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) \\
&+ \left(1 + \frac{\beta L}{f}y'\right)u' = -\frac{gH}{fUL}Z'_{y'}
\end{aligned} \qquad (12.24)$$

and

$$\left(1 + \frac{HZ'}{\bar{Z}}\right)\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \frac{HL}{\bar{Z}TU}\frac{\partial Z'}{\partial t'} + \frac{H}{\bar{Z}}\left(u'\frac{\partial Z'}{\partial x'} + v'\frac{\partial Z'}{\partial y'}\right) = 0. \quad (12.25)$$

We wish to capitalize on the following to simplify our equations:

(i) The dominance of the Coriolis force;

(ii) The approximate validity of geostrophy;

(iii) The small excursions of f from its mean value.

Item (ii) leads to taking

$$fU = g\frac{H}{L}$$

or

$$H = \frac{fUL}{g}. \tag{12.26}$$

Item (i) leads to our taking

$$R_0 = \frac{U}{fL} \ll 1 \tag{12.27}$$

and

$$R_{0T} = \frac{1}{fT} \ll 1. \tag{12.28}$$

For *simplicity* we will take

$$R_0 = R_{0T} = \epsilon. \tag{12.29}$$

Item (iii) leads us to write

$$\frac{\beta L}{f} = \epsilon\beta'. \tag{12.30}$$

We may now rewrite (12.26) as

$$H = \epsilon\frac{f^2L^2}{g},$$

in which case the non-dimensional parameters in (12.25) become

$$\frac{H}{\bar{Z}} = \epsilon\frac{f^2L^2}{g\bar{Z}}$$

and

$$\frac{H}{\bar{Z}}\frac{L}{TU} = \epsilon\frac{f^2L^2}{g\bar{Z}}\frac{R_{0T}}{R_0} = \epsilon\frac{f^2L^2}{g\bar{Z}}.$$

12.3.1 Rossby radius

Now let us define a distance R by the following relation

$$\frac{f^2L^2}{g\bar{Z}} = \frac{L^2}{R^2}$$

or

$$R^2 = \frac{g\bar{Z}}{f^2};$$

R is known as the Rossby radius.

12.3.2 Rossby number expansion

We will take

$$L^2 = R^2. \tag{12.31}$$

As a result of the above, we may rewrite (12.23)–(12.25) as follows:

$$\epsilon\left(\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) - (1 + \epsilon\beta' y')v' = -Z'_{x'} \tag{12.32}$$

$$\epsilon\left(\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) - (1 + \epsilon\beta' y')u' = -Z'_{y'} \tag{12.33}$$

and

$$(1 + \epsilon Z')\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \epsilon\left(\frac{\partial Z'}{\partial t'} + u'\frac{\partial Z'}{\partial x'} + v'\frac{\partial Z'}{\partial y'}\right) = 0. \tag{12.34}$$

We now expand all our variables in powers of ϵ :

$$\begin{aligned} u' &= u_0 + \epsilon u_1 + \ldots \\ v' &= v_0 + \epsilon v_1 + \ldots \\ Z' &= Z_0 + \epsilon Z_1 + \ldots \ . \end{aligned} \tag{12.35}$$

We next substitute (12.35) into (12.32)–(12.34) and order our equations by powers of ϵ. At zeroth order we have

$$-v_0 = -Z_{0,x'} \tag{12.36}$$

$$u_0 = -Z_{0,y'} \tag{12.37}$$

and consistent with (12.36) and (12.37)

$$\frac{\partial u_0}{\partial x'} + \frac{\partial v_0}{\partial y'} = 0. \tag{12.38}$$

Equations 12.36 and 12.37 are simply the geostrophic relations and as such they tell us nothing about time evolution. Equation 12.38 tells us that horizontal divergence is $O(\epsilon)$.

At first order in ϵ we have

$$\frac{\partial u_0}{\partial t'} + u_0\frac{\partial u_0}{\partial x'} + v_0\frac{\partial u_0}{\partial y'} - v_1 - \beta' y' v_0 = -Z_{1,x'} \tag{12.39}$$

$$\frac{\partial v_0}{\partial t'} + u_0\frac{\partial v_0}{\partial x'} + v_0\frac{\partial v_0}{\partial y'} + u_1 - \beta' y' u_0 = -Z_{1,y'} \tag{12.40}$$

$$\frac{\partial u_1}{\partial x'} + \frac{\partial v_1}{\partial y'} + \left(\frac{\partial Z_0}{\partial t'} + u_0\frac{\partial Z_0}{\partial x'} + v_0\frac{\partial Z_0}{\partial y'}\right) = 0. \tag{12.41}$$

We next differentiate (12.39) with respect to y', and (12.40) with respect to x', and subtract the results just as in Section 12.2 to obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t'}\right. &+ \left. u_0\frac{\partial}{\partial x'} + v_0\frac{\partial}{\partial y'}\right)\left(\frac{\partial v_0}{\partial x'} - \frac{\partial u_0}{\partial y'} + \beta' y'\right) + \left(\frac{\partial u_1}{\partial x'} + \frac{\partial v_1}{\partial y'}\right) \\ &+ \beta' y' \underbrace{\left(\frac{\partial v_0}{\partial y'} + \frac{\partial u_0}{\partial x'}\right)}_{=0} + \underbrace{\left(\frac{\partial u_0}{\partial x'} + \frac{\partial v_0}{\partial y'}\right)}_{=0}\left(\frac{\partial v_0}{\partial x'} - \frac{\partial u_0}{\partial y'}\right) = 0.\end{aligned} \tag{12.42}$$

Using (12.41) we finally get[2]

$$\left.\frac{D}{Dt}\right|_0 \left(\frac{\partial v_0}{\partial x'} - \frac{\partial u_0}{\partial y'} + \beta' y'\right) - \left.\frac{D}{Dt}\right|_0 Z_0 = 0. \tag{12.43}$$

Equation 12.43 is simply (12.11) where the advective velocities and the relative vorticity are evaluated geostrophically. Equations 12.36, 12.37, and 12.43 completely determine the zeroth order fields, but note that we had to go to first order in ϵ to get (12.43). Not surprisingly, the evolution of quasi-geostrophic flow is completely determined by the vorticity equation. (Note that $|f| \gg |\zeta|$.)

12.4 Quasi-geostrophy in a stratified, compressible atmosphere

Given the close relation we have noted in Chapter 11 between the shallow water equations and the equations for internal waves in a deep atmosphere, we may reasonably anticipate that the quasi-geostrophic equations for a deep atmosphere will be similar to those we have just obtained.

Our equations of motion in log $-p$ coordinates are

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w^*\frac{\partial}{\partial z^*}\right) u - fv = -\Phi_x \tag{12.44}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w^*\frac{\partial}{\partial z^*}\right) v + fu = -\Phi_y \tag{12.45}$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \epsilon^{z^*}\frac{\partial}{\partial z^*}(e^{-z^*}w^*) = 0 \tag{12.46}$$

$$\frac{\partial \Phi}{\partial z^*} = RT \tag{12.47}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) T + w^*\left(\frac{\partial T}{\partial z^*} + \frac{RT}{c_p}\right) = 0. \tag{12.48}$$

If

[2]The notation $\left.\frac{D}{Dt}\right|_0$ refers to the standard substantial derivative where the advecting velocities are evaluated geostrophically.

$$
\begin{aligned}
w^* &= Ww' \\
u, v &= Uu', Uv' \\
z^* &= Hz' \\
x, y &= Lx', Ly'
\end{aligned}
$$

then we know from Section 12.3 that

$$W \sim \frac{H}{L} U \epsilon$$

because the geostrophic divergence ~ 0. Hence the vertical advections will be at least a factor ϵ smaller than the horizontal advections. However, the latter are already $O(\epsilon)$ compared to the Coriolis term. Thus *vertical advections will not enter our equations* at either zeroth or first order in ϵ, and Equations 12.44 and 12.45 are essentially *identical to our shallow water equations* for u and v. Thus at zeroth order

$$-f_0 v_G = -\Phi_x \tag{12.49}$$

$$-f_0 u_G = -\Phi_x \tag{12.50}$$

(where $f = f_0 + \beta y$). Similarly, to first order

$$\left(\frac{\partial}{\partial t} + u_G \frac{\partial}{\partial x} + v_G \frac{\partial}{\partial y} \right) \left(\frac{\partial v_G}{\partial x} - \frac{\partial u_G}{\partial y} + f \right) + f_0 \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = 0. \tag{12.51}$$

(N.B. we are retaining dimensional variables.)

Equation 12.46 relates w^* to

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}.$$

Equation 12.47 allows us to rewrite (12.48) as

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \frac{\partial \Phi}{\partial z^*} + w^* R \left(\frac{\partial T}{\partial z^*} + \frac{RT}{c_p} \right) = 0. \tag{12.52}$$

By analogy with our shallow water analysis we can, to lowest order, replace

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

in (12.52) with

$$\frac{\partial}{\partial t} + u_G\frac{\partial}{\partial x} + v_G\frac{\partial}{\partial y}.$$

Also, we can replace

$$\frac{\partial T}{\partial z^*} + \frac{RT}{c_p}$$

with its horizontal average

$$\frac{\partial \bar{T}}{\partial z^*} + \frac{R\bar{T}}{c_p}$$

(Why?).
Thus we have

$$w^* = \frac{(\frac{\partial}{\partial t} + u_G\frac{\partial}{\partial x} + v_G\frac{\partial}{\partial y})\frac{\partial \Phi}{\partial z^*}}{R(\frac{d\bar{T}}{dz^*} + \frac{R\bar{T}}{c_p})}. \qquad (12.53)$$

Equations 12.53 and 12.46 then give

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = e^{z^*}\frac{\partial}{\partial z^*}\left\{\frac{e^{-z^*}(\frac{\partial}{\partial t} + u_G\frac{\partial}{\partial x} + v_G\frac{\partial}{\partial y})\frac{\partial \Phi}{\partial z^*}}{R(\frac{d\bar{T}}{dz^*} + \frac{R\bar{T}}{c_p})}\right\}. \qquad (12.54)$$

If we let

$$S = R\left(\frac{d\bar{T}}{dz^*} + \frac{R\bar{T}}{c_p}\right),$$

(12.54) becomes

$$\begin{aligned}\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \quad &= \quad \left(\frac{\partial}{\partial t} + u_G\frac{\partial}{\partial x} + v_G\frac{\partial}{\partial y}\right)\left\{e^{z^*}\frac{\partial}{\partial z^*}\left(\frac{e^{-z^*}}{S}\frac{\partial \Phi}{\partial z^*}\right)\right\} \\ &\quad + \frac{1}{S}\underbrace{(\frac{\partial u_G}{\partial z^*}\frac{\partial}{\partial x} + \frac{\partial v_G}{\partial z^*}\frac{\partial}{\partial y})\frac{\partial \Phi}{\partial z^*}}_{=0 \ \textit{by geostrophy}}. \qquad (12.55)\end{aligned}$$

With (12.55), (12.51) becomes

$$\left(\frac{\partial}{\partial t} + u_G\frac{\partial}{\partial x} + v_G\frac{\partial}{\partial y}\right)\left(\frac{\partial u_G}{\partial x} - \frac{\partial u_G}{\partial y} + f\right)$$

$$+ \left(\frac{\partial}{\partial t} + u_G\frac{\partial}{\partial x} + v_G\frac{\partial}{\partial y}\right)\left(e^{z^*}\frac{\partial}{\partial z^*}\left(\frac{f}{S}e^{-z^*}\frac{\partial \Phi}{\partial z^*}\right)\right) = 0$$

or

$$\left(\frac{\partial}{\partial t} + u_G\frac{\partial}{\partial x} + v_G\frac{\partial}{\partial y}\right)$$

$$\left\{\frac{\partial v_G}{\partial x} - \frac{\partial u_G}{\partial y} + f + e^{z^*}\frac{\partial}{\partial z^*}\left(\frac{f_0}{S}e^{-z^*}\frac{\partial \Phi}{\partial z^*}\right)\right\} = 0. \qquad (12.56)$$

12.4.1 Pseudo potential vorticity

The quantity in brackets in Equation 12.56 is called the *pseudo-potential vorticity* since it is conserved not on particle trajectories but on their horizontal projections. The relation between (12.56) and (12.43) is much what we would expect from our earlier comparison of the equations for shallow water waves and internal waves. Using (12.49) and (12.50), (12.56) becomes

$$\left(\frac{\partial}{\partial t} - \frac{1}{f_0}\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} + \frac{1}{f_0}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y}\right)\left\{\frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) + f\right.$$

$$\left. + e^{z^*}\frac{\partial}{\partial z^*}\left(\frac{f_0}{S}e^{-z^*}\frac{\partial \Phi}{\partial z^*}\right)\right\} = 0. \qquad (12.57)$$

The quasi-geostrophic approximation was originally developed by Charney (1948). Note that the height field completely determines quasi-geostrophic motion – even its time evolution.