Chapter 10

Variable Basic States

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Supplemental reading:

Lindzen (1981)

Lindzen and Holton (1968)

Plumb (1977)

10.1 WKBJ analysis

Most waves in the atmosphere and ocean have phase speeds of the same order as the basic flow. In contrast to tides, variations in the basic flow will be of major importance to such waves. Moreover, waves can strongly affect mean flows. Again, we will choose to study this situation in the simplest relevant configuration. We will ignore rotation and retain log –p coordinates. Our basic flow will depend only on z^*, and we will consider perturbations for which $v' = 0$. Our equations are

$$\frac{\partial u'}{\partial t} + U_0(z^*)\frac{\partial u'}{\partial x} + w^*\frac{dU_0}{dz^*} + \frac{\partial \Phi'}{\partial x} = -au' \tag{10.1}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w^*}{\partial z^*} - w^* = 0 \tag{10.2}$$

$$\frac{\partial \Phi'}{\partial z^*} = RT' \tag{10.3}$$

$$\frac{\partial T'}{\partial t} + U_0 \frac{\partial T'}{\partial x} + w^* \left(\frac{dT_0}{dz^*} + \kappa T_0 \right) = -aT', \tag{10.4}$$

where $a =$ linear damping rate. If we look for solutions of the form $e^{ik(x-ct)}$, 'a' can simply be taken as a modification of 'c', so for the moment we will simply forget the damping. Assuming harmonic dependence on x and t, Equations 10.1–10.4 become

$$-ik(c - U_0)u' + w^* \frac{dU_0}{dz^*} + ik\Phi' = 0 \tag{10.5}$$

$$iku' + \frac{dw^*}{dz^*} - w^* = 0 \tag{10.6}$$

$$-ik(c - U_0)\frac{d\Phi'}{dz^*} + w^* R \left(\frac{dT_0}{dz^*} + \kappa T_0 \right) = 0. \tag{10.7}$$

Eliminating u' and Φ' yields

$$(c - U_0) \left\{ (c - U_0) \left(\frac{d^2 w^*}{dz^{*2}} - \frac{dw^*}{dz^*} \right) + w^* \frac{dU_0}{dz^*} + w^* \frac{d^2 U_0}{dz^{*2}} \right\} + w^* R \left(\frac{dT_0}{dz^*} + \kappa T_0 \right) = 0. \tag{10.8}$$

Introducing $\tilde{w} = w^* e^{-z^*/2}$ again and organizing terms we get

$$\frac{d^2 \tilde{w}}{dz^{*2}} + \left\{ \underbrace{\frac{R(\frac{dT_0}{dz^*} + \kappa T_0)}{(c - U_0)^2}}_{A} + \underbrace{\frac{(\frac{d^2 U_0}{dz^{*2}} + \frac{dU_0}{dz^*})}{(c - U_0)}}_{B} - \frac{1}{4} \right\} \tilde{w} = 0. \tag{10.9}$$

Only term B is new; term A differs from earlier results only in that a constant c has been replaced by a variable $c - U_0(z^*)$. If $U_0(z^*)$ is varying 'slowly' then we may ignore B (though it will play a crucial rôle in certain instability problems). Let

$$\lambda^2 \equiv \left\{ \frac{R(\frac{dT_0}{dz^*} + \kappa T_0)}{(c - U_0)^2} - \frac{1}{4} \right\}. \tag{10.10}$$

When λ^2 is constant the solution of (10.9) is trivial. When it is varying slowly we have an asymptotically approximate solution in the form of the WKBJ approximation that is almost as trivial[1]:

$$w^* \approx \frac{Ae^{z^*/2}}{\lambda^{1/2}} \exp\left\{ -i \int^{z^*} \lambda dz^* \right\}. \tag{10.11}$$

The replacement of $i\lambda z^*$ by $i\int^{z^*} \lambda dz^*$ is intuitively obvious. The slowly varying amplitude factor, $\lambda^{-1/2}$ will be discussed later. Equation 10.11 gives us a basis for assessing the effects of the basic state on a wave. We have already noticed the profound effect of density stratification in producing exponential growth with height. *This almost certainly accounts for the increasing prominance of internal gravity waves in the upper atmosphere.* For the moment, however, we wish to concentrate on the changes produced by variations in U_0. If we concentrate on large values of λ^2,

$$\lambda^2 \approx \frac{R(\frac{dT_0}{dz^*} + \kappa T_0)}{(c - U_0)^2} = \frac{H^2 N^2}{(c - U_0)^2} \tag{10.12}$$

$$\lambda \approx \pm \frac{HN}{(c - U_0)} \tag{10.13}$$

$$\lambda^{-1/2} \approx \left| \frac{(c - U_0)}{HN} \right|^{1/2}. \tag{10.14}$$

As we have already noted, the local VWL approaches zero as $(c - U_0) \to 0$. At the same time, w^* also $\to 0$ as $(c - U_0) \to 0$.

What about other fields? From (10.6) we have

$$u' = \frac{i}{k} \left(\frac{dw^*}{dz^*} - w^* \right). \tag{10.15}$$

From (10.7) we have

[1]Our approach here follows Lindzen (1981).

$$T' = -\frac{i}{k}\frac{H^2N^2}{c-U_0}w^*. \tag{10.16}$$

From (10.5)

$$\Phi' = \frac{i}{k}(c-U_0)\left(\frac{dw^*}{dz^*} - w^*\right) + w^*\frac{dU_0}{dz^*}\frac{i}{k}. \tag{10.17}$$

We also want w in terms of w^*. Now

$$w^* = -\frac{1}{p_0}\left(\underbrace{\frac{\partial p'}{\partial t} + U_0\frac{\partial p'}{\partial x}}_{ik(U_0-c)p'=ik\rho_0(U_0-c)\Phi'} + w\underbrace{\frac{dp_0}{dz}}_{-\frac{p_0}{H}w}\right)$$

(*viz.* Chapters 4 and 9).
So, using (10.17),

$$-p_0w^* = \rho_0(c-U_0)\left\{(c-U_0)\left(\frac{dw^*}{dz^*} - w^*\right) + w^*\frac{dU_0}{dz^*}\right\} - \frac{p_0}{H}w$$

or

$$w = Hw^* + \frac{1}{g}(c-U_0)\left\{(c-U_0)\left(\frac{dw^*}{dz^*} - w^*\right) + w^*\frac{dU_0}{dz^*}\right\}. \tag{10.18}$$

Assuming large λ^2 again

$$u' \approx \frac{i}{k}(-i\lambda)w^{*\prime} \approx \frac{A}{k}\left|\frac{HN}{(c-U_0)}\right|^{1/2} e^{z^*/2}e^{-i\int^{z^*}\lambda dz^*} \tag{10.19}$$

$$T' \approx -\frac{i}{k}\frac{H^2N^2}{|c-U_0|^{1/2}}\frac{A}{|HN|^{1/2}}e^{z^*/2}e^{-i\int^{z^*}\lambda dz^*} \tag{10.20}$$

$$\Phi' \approx \frac{i}{k}(c-U_0)(-i\lambda)w^* \approx \frac{HN}{k}Ae^{z^*/2}\left|\frac{c-U_0}{HN}\right|^{1/2} e^{-i\int^{z^*}\lambda dz^*} \tag{10.21}$$

$$w' \approx Hw^* \approx AH\left|\frac{c-U_0}{HN}\right|^{1/2} e^{z^*/2}e^{-i\int^{z^*}\lambda dz^*}. \tag{10.22}$$

Thus, while w' and Φ' go to zero as $c - U_0 \to 0, u'$ and T' blow up. Is this consistent with the second Eliassen-Palm theorem?

$$\underbrace{\rho_0 \overline{u'w'}}_{p_0 = \rho_0 g H} = \frac{p_0}{gH}\frac{1}{2}|u'||w'|$$

since (10.19) and (10.2) imply u' and w' are in phase, and $\frac{1}{\pi}\int_0^\pi \sin^2\phi\, d\phi = \frac{1}{2}$. From (10.19) and (10.20) we have

$$\begin{aligned} \rho_0 \overline{u'w'} &= \frac{p_0(0)e^{-z^*}}{gH}\frac{1}{2}\frac{A}{k}e^{z^*/2} \cdot AHe^{z^*/2} \\ &= \frac{1}{2}\frac{A^2}{gk}p_0(0) = \text{ constant.} \end{aligned}$$

The answer, therefore, is yes. It is interesting to note that the factor $\lambda^{-1/2}$ in Equation 10.11 is crucial in this regard. The momentum flux $\rho_0\overline{u'w'}$ is actually the Wronskian of the wave equation, and the factor $\lambda^{-1/2}$ guarantees that the Wronskian of the WBKJ solution remains constant.

10.2 Critical level behavior

We may next ask what happens when $c - U_0$ actually goes through zero. Clearly something must happen. Eliassen and Palm's first theorem tells us that if a wave travels through a critical layer, $\rho_0\overline{u'w'}$ must change sign. This implies an exchange of momentum flux carried by the wave with the mean flow at the critical level. The only possible alternative is that the wave is totally reflected at the critical level, in which case $\overline{p'w'} = \rho_0\overline{u'w'} = 0$.

An answer, for linear theory in the limit of vanishing damping, was obtained by Booker and Bretherton (1967). Let $z^* = 0$ be the critical level and let

$$U_0 = c + \frac{dU_0}{dz^*}z^*$$

in the neighbourhood of $z^* = 0$. Equation 10.9 becomes

$$\frac{d^2\tilde{w}}{dz^{*2}} + \left\{ \underbrace{\frac{H^2N^2}{(\frac{dU_0}{dz^*})^2 z^{*2}}}_{this\ term\ dominates} + \frac{\frac{dU_0}{dz^*}}{-(\frac{dU_0}{dz^*})z^*} - \frac{1}{4} \right\} \tilde{w} = 0,$$

which, in the neighbourhood of $z^* = 0$, is approximated by

$$\frac{d^2\tilde{w}}{dz^{*2}} + \frac{H^2N^2}{(\frac{dU_0}{dz^*})^2 z^{*2}} \tilde{w} = 0. \tag{10.23}$$

Equation 10.23 is simply a special case of Euler's equation, whose solution is

$$\tilde{w} = Az^{*\nu}, \tag{10.24}$$

where

$$\nu(\nu - 1) + \underbrace{\frac{H^2N^2}{(\frac{dU_0}{dz^*})^2}}_{=\frac{N^2}{(\frac{dU_0}{dz})^2} \equiv Ri} = 0 \tag{10.25}$$

or

$$\nu = \frac{1 \pm \sqrt{1 - 4Ri}}{2}. \tag{10.26}$$

10.2.1 Richardson number

The non-dimensional number, Ri, is known as the *Richardson Number*. The nature of our solution will depend greatly on whether $Ri \lessgtr 1/4$. Now Ri is a measure of how rapidly $(c - U_0)$ is varying and for WKBJ theory to be appropriate we would want $Ri > 1/4$. In the atmosphere Ri is typically 1–10. In this case (10.24) becomes

$$\tilde{w} = |z^*|^{1/2} e^{\pm i\mu \ln z^*}, \tag{10.27}$$

where

$$\mu = \sqrt{Ri - 1/4},$$

and the + sign is appropriate to upward propagation (show this for yourself). Note that (10.27) is essentially our WKBJ solution in the neighbourhood of $z^* = 0$. Note also that this is very definitely not the case when $Ri < 1/4$! (It is not irrelevant to note that when $Ri < 1/4$ the fluid can be and often is unstable; i.e., waves draw energy from the basic state.)

10.2.2 Conditions for absorption

For $z^* > 0$, (10.27) becomes

$$\tilde{w} = z^{*1/2} e^{i\mu \ln z^*}. \tag{10.28}$$

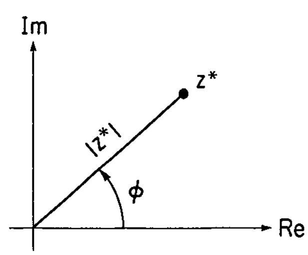
Equation 10.28 has a branch point at $z^* = 0$. In continuing z^* to negative values, (10.28) does not lead to a unique answer. The answer depends on whether we pass above or below $z^* = 0$ in traversing the complex z^*-plane[2].

Now, had we retained damping

$$\begin{aligned} -ik(c - U_0) \rightarrow \quad - \quad & ik(c - U_0) + a \\ & = -ik(\tfrac{ai}{k} - \tfrac{dU_0}{dz^*} z^*). \end{aligned}$$

[2]For those who don't remember what a branch point is, here are a few explanatory remarks.

Consider z^* in the complex plane



We may write $z^* = | z^* | e^{i\phi}$. Then

$$\begin{aligned} \ln z^* &= \ln(|z^*| e^{i\phi}) \\ &= \ln |z^*| + \ln e^{i\phi} \\ &= \ln |z^*| + i\phi. \end{aligned}$$

Now, ϕ in the above diagram is arbitrary to a positive or negative multiple of 2π. The choice of branch is tantamount to the choice of the appropriate 2π range for ϕ. If we are moving across the origin along the real axis, we must, in this connection, decide whether ϕ is going from 0 on the positive side to π on the negative side or from 0 on the positive side to $-\pi$ on the negative side. The former corresponds to passing above the origin; the latter to passing below the origin.

The branch point is now

$$z^* = \frac{\frac{ai}{k}}{\frac{dU_0}{dz^*}}, \text{ Im } z^* > 0.$$

So with damping the branch point moves above $z^* = 0$, and hence in going from positive to negative z^* we pass below the branch point; that is, z^* goes to $|z^*|e^{-i\pi}$, and (10.28) goes to

$$w = -i|z^*|^{1/2}e^{i\mu(\ln|z^*|-i\pi)} = -i|z^*|^{1/2}e^{\mu\pi}e^{i\mu\ln|z^*|}. \tag{10.29}$$

It is easily shown from (10.28) and (10.29) that if $\rho_0\overline{u'w'} = A$ below $z^* = 0$, it will equal $-Ae^{-2\mu\pi}$ above. For $Ri > 1$, this implies almost complete *wave absorption* at the critical level. This is consistent with our earlier observation that group velocity $\rightarrow 0$ as $c - U_0 \rightarrow 0$, thus allowing any damping enough time to act.

10.2.3 Linear and nonlinear limits

By now you may be wondering whether we may not be pushing matters too far: solutions which are blowing up and finite momentum fluxes deposited in infinitesimal layers. At the very least, solutions blowing up might suggest that linearity is in question. For solutions in the neighbourhood of the critical level let us examine the relative magnitudes of linear and nonlinear terms. We have

$$w^* \approx A|z^*|^{1/2}e^{i\mu\ln|z^*|} \underbrace{e^{z^*/2}}_{\textit{incorporate this into } A}.$$

Let us, for simplicity, assume $\mu \gg 1$. Then

$$\frac{dw^*}{dz^*} \approx A\sqrt{-z^*}\left(\frac{-i\mu}{z^*}\right)e^{i\mu\ln(-z^*)}.$$

Also

$$iku' \approx -\frac{dw^*}{dz^*}$$

$$\begin{aligned} u' &\approx \frac{i}{k} A\sqrt{-z^*}\left(\frac{-i\mu}{z^*}\right) e^{i\mu \ln(-z^*)} \\ &\approx \frac{A\mu}{k}\frac{1}{\sqrt{-z^*}} e^{i\mu \ln(-z^*)} \end{aligned}$$

and

$$\begin{aligned} \frac{du'}{dz^*} &\approx \frac{A\mu}{k}\frac{1}{\sqrt{-z^*}}\left(\frac{-i\mu}{z^*}\right) e^{i\mu \ln(-z^*)} \\ &\approx \frac{Ai\mu^2}{k}\frac{1}{(-z^*)^{3/2}} e^{i\mu \ln(-z^*)}. \end{aligned}$$

Typical nonlinear terms in the horizontal momentum equation are $u'\frac{\partial u'}{\partial x}$ and $w'\frac{\partial u'}{\partial z}$.

$$\begin{aligned} |u'\frac{\partial u'}{\partial x}| &\approx \frac{A^2\mu^2}{k}\frac{1}{|z^*|} \\ |w^*\frac{\partial u'}{\partial z^*}| &\approx \frac{A^2\mu^2}{k}\frac{1}{|z^*|}. \end{aligned}$$

Typical linear terms are $ik(U_0 - c)u'$ and $w^*\frac{dU_0}{dz^*}$.

$$\begin{aligned} |w^*\frac{dU_0}{dz^*}| &\approx A|z^*|^{1/2}\frac{dU_0}{dz^*} \\ |ik(U_0 - c)u'| &\approx A\mu\frac{dU_0}{dz^*}|z^*|^{1/2}. \end{aligned}$$

(Note the latter is bigger.) The ratio of linear to nonlinear terms is

$$\left|\frac{\text{Linear}}{\text{Nonlinear}}\right| \approx \left|\frac{A\mu\frac{dU_0}{dz^*}|z^*|^{1/2}}{\frac{A^2\mu^2}{k}|z^*|^{-1}}\right| \approx \frac{k\frac{dU_0}{dz^*}}{\mu A}(-z^*)^{3/2}.$$

At that z^* for which this ratio ≈ 1 we expect nonlinear effects to be important. Now $\mu \sim Ri^{1/2}(Ri = RS/(\frac{dU_0}{dz^*})^2)$. The ratio ≈ 1, when

$$(-z^*_{NL}) \sim (\frac{RiA}{k(RS)^{1/2}})^{2/3}. \tag{10.30}$$

Not surprisingly z^*_{NL} depends on the wave amplitude. Now nonlinear problems are generally difficult. Can damping help us avoid these difficulties (and, at the same time, preserve our earlier linear results on what happens at a critical level).

If we keep the linear damping shown in Equations 10.1 and 10.4, it is equivalent to replacing $ik(U_0 - c)$ with $ik(U_0 - c) + a$, or replacing $(c - U_0)$ with $(c - U_0) + \frac{ai}{k}$. Damping will begin to dominate when

$$\frac{a}{k} \sim (c - U_0)$$

or

$$\frac{a}{k} \sim \left| \frac{dU_0}{dz^*} z^* \right|$$

or when

$$z^* = z^*_d \approx -\frac{a}{k\frac{dU_0}{dz}} \approx -\frac{a}{k}\left(\frac{Ri}{RS}\right)^{1/2}. \tag{10.31}$$

When $|z^*_d| > |z^*_{nl}|$, nonlinearity will not have a chance to dominate and our earlier linear analysis will be appropriate – except that wave absorption will take place over a finite layer.

10.3 Damping and momentum deposition

The effect of damping is really more extensive than the above argument suggests. In the presence of damping, (10.13) becomes

$$\lambda \approx \frac{HN}{(c - U_0) + \frac{ai}{k}} \approx \frac{NH(c - U_0)}{(c - U_0)^2 + \frac{a^2}{k^2}} - \frac{NH(\frac{a}{k})i}{(c - U_0)^2 + \frac{a^2}{k^2}}$$

$$\approx \lambda_r - i\lambda_i. \tag{10.32}$$

λ now has an imaginary component which produces exponential decay – in addition to propagation. (N.B. we have decay because of our choice of a minus sign in (10.11).) This decay means that for a given wave forcing below the critical level, the effective values of A (*viz.*, (10.30)) near the critical level will be diminished – and the chances of damping dominating are increased.

How these matters actually work out is still a matter of some controversy. For a problem we will soon look at, the gravity wave periods will be rather long and radiative cooling will provide damping sufficient to obviate nonlinear effects. (In this case damping appears in (10.4), but not in (10.1). As an exercise you will show that this does not qualitatively alter the above results.) Other cases are not so clear. Benney and Bergeron (1969) have shown that the nonlinear limit leads to reflection rather than absorption at the critical level. However, the nonlinear solution is unstable and may lead to turbulence which in turn may lead to absorption again.

10.3.1 Violation of the second Eliassen-Palm theorem

We now come to an obvious – but major – point. When we have damping, the second Eliassen-Palm theorem is violated. Instead, we have

$$\rho_0\overline{u'w'} = (\rho_0\overline{u'w'})_0 e^{-2\int_0^{z^*}\lambda_i dz^*}, \tag{10.33}$$

where $z^* = 0$ is assumed not to be a critical level, and

$$\frac{d}{dz}\rho_0\overline{u'w'} = \frac{1}{H}\frac{d}{dz^*}\rho_0\overline{u'w'} = \frac{-2\lambda_i}{H}\rho_0\overline{u'w'}. \tag{10.34}$$

The response of the mean flow (in the absence of rotation) will be given by

$$\frac{\partial\bar{u}}{\partial t} - \underbrace{\nu}_{\text{viscosity}}\frac{\partial^2\bar{u}}{\partial z^2} = -\frac{2\lambda_i}{H\rho_0}(\rho_0\overline{u'w'})_0 e^{-2\int_0^{z^*}\lambda_i dz^*}, \tag{10.35}$$

that is, wave absorption will lead to mean flow acceleration. It is also important to note that absorption increases markedly as $\bar{u}$ approaches c. (It is worth noting that in the presence of rotation, the response of $\bar{u}$ is no longer so clear. The Coriolis term can balance part of the Reynolds stress divergence, and the change in $\bar{u}$ might be negligible.) This mechanism is believed to play a major role in generating the quasi-biennial oscillation of the tropical stratosphere.

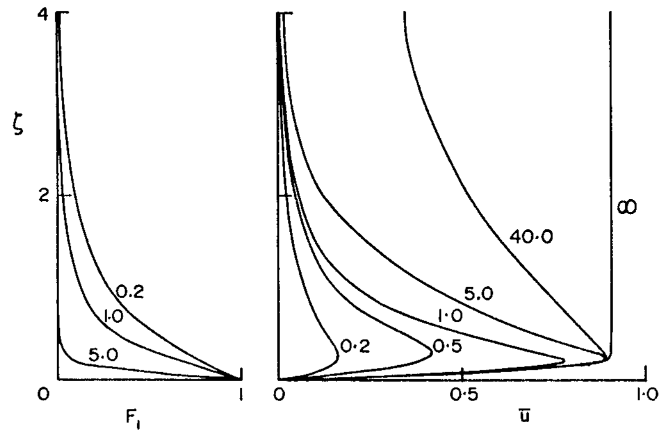


Figure 10.1: Profiles of mean velocity $\bar{u}$ and wave momentum flux F for single-wave forcing. Curves are labelled with time τ.

10.4 Quasi-biennial oscillation

Let us briefly recall what this phenomenon is. Its main features are shown in Figure 5.17. Easterly and westerly regimes in the equatorial stratosphere (of magnitude 20 m/s) alternate quasi-periodically with an average period of about twenty six months. There is also an apparent downward phase propagation. Now, how do gravity waves enter this picture? Remember, for vertical propagation to be possible gravity waves must have periods short compared to the pendulum day. For most of the earth this means periods shorter than a day or so, and from what we saw in Chapter 5, most eddy energy is at periods longer than this. So, over most of the earth we do not anticipate that the bulk of the eddies will behave as gravity waves. However, as we approach the equator, the pendulum day goes to infinity, and the energy in these long periods can begin to propagate vertically as internal gravity waves. Also, the Coriolis term legitimately drops out of (10.35). It is, in fact, observed that such waves are produced in the neighbourhood of the equator – with both easterly and westerly phase speeds. The exact nature of their excitation is not yet fully understood, but it appears to originate near the tropical tropopause (*ca.* 16 km). Let us take the existence of such waves as given.

Figure 10.1 schematically illustrates how a fluid at rest (initially) might respond to a wave with phase speed $c = 1$. $\bar{u}$ is accelerated towards 1, but note that as $\bar{u}$ is brought towards 1 in the lower part of the domain, λ_i must increase and this prevents the wave from acting on the upper part of the domain. In the calculations upon which Figure 10.1 is based, diffusion eventually carries momentum upward. By the time a sharp shear zone develops near the bottom, the fluid above this shear zone is pretty much isolated from the westerly wave. If, however, there were also an easterly wave with $c = -1$, it would be free to propagate upward accelerating $\bar{u}$ towards -1. The downward descent of the easterly shear zone (on top of the previous westerly shear zone) would produce severe stresses which would wipe out the westerly zone and allow the westerly waves to propagate upward again starting the whole process anew. Of course, the actual calculations have some nuances which are here omitted. But, the essential mechanism is as described. In particular, the *amplitude* of the oscillation in $\bar{u}$ is determined by the *phase speeds* of the upward propagating waves, and the *period* of the oscillation in $\bar{u}$ is determined by the *average intensity* of the upward propagating waves.

This mechanism was first described by Lindzen and Holton (1968), and brought to its current form by Lindzen (1971) and Holton and Lindzen (1972). A highly simplified version of the theory appropriate to a laboratory configuration was developed by Plumb (1977) and the mechanism has, in fact, been simulated in the laboratory by Plumb and McEwan (1978).