**Stress in the Ocean**

The treatment of stress and strain in fluids is complicated and somewhat beyond the scope of this class. Those really interested should look into this further in Batchelor, *Introduction to Fluid Dynamics*, given as a reference at the beginning of the class. The scaling up of turbulent stresses to oceanic scales is even more esoteric. Here we will merely summarize what is known about the form of the "body force", $F$, for oceanic flows. It involves a tensor with three components of stress for each of the three momentum equations.

$$F^x = \left[ \tau_x^x + \tau_y^x + \tau_z^x \right]$$
$$F^y = \left[ \tau_x^y + \tau_y^y + \tau_z^y \right]$$
$$F^z = \left[ \tau_x^z + \tau_y^z + \tau_z^z \right]$$

The notation is that the superscript refers to the stress direction of the component and the subscript refers to the derivative of the component. The relationship of stress to strain (fluid response) can be complicated. For fluids like water and for scales typical of those we will be discussing in this class, it can take on the following form:

$$\tau_j^i = \rho A^j \frac{\partial u^i}{\partial x_j}$$

where $A^j$ is a constant known as the "eddy viscosity". For fluids that are shallow relative to their widths, these three assumed constants ($A^x$, $A^y$, $A^z$) are not the same: the vertical component is much smaller than the two horizontal components, which are usually (not always) assumed to be equal. If we insert this definition into the above equation for the vector body force, we can get the following relationship (where $A^x = A^y = A^h$)

$$F^x = \rho \left[ A^h u_{xx} + A^h u_{yy} + A^z u_{zz} \right]$$
$$F^y = \rho \left[ A^h v_{xx} + A^h v_{yy} + A^z v_{zz} \right]$$
$$F^z = \rho \left[ A^h w_{xx} + A^h w_{yy} + A^z w_{zz} \right]$$

Our dynamical equations have already incorporated one of these body forces into them: it is the third component of each of the above shear stresses. Consider the vertical integral of one of these three forces (say the first one).

$$\int_{-H}^{0} dz F^x = \int dz [\tau_x^x + \tau_y^x + \tau_z^x] \approx \frac{\partial}{\partial x} \int dz \tau^x + \frac{\partial}{\partial y} \int dz \tau^x + \tau^x(z=0) - \tau^x(z=-H)$$

The approximation is because we are taking $H$ to be a constant depth level and assuming the surface to be flat (or it's variation $h$ to be small compared to the depth of the ocean). Consider now the last two terms in the above. The last one is proportional to the stress on the bottom and the preceeding one is due to the stress on the top or free surface. If we write the bottom stress as

$$\tau^x(z=-H) = \rho A^z \frac{\partial u}{\partial z}(z=-H) = \rho r u ,$$

we see that we can now associate this term with the simple bottom friction we have retained from our "pucks on ice" model. It is this term that plays an important role in the Stommel model for the western intensification of the wind-driven gyres. As for the wind driving, this is the just the surface component of the stress $\tau^x(z=0)$. For the Sverdrup balance, we obtained the following equation

$$\beta v = \frac{1}{\rho}(\nabla_h \times \bar{F}), or$$

$$\beta \int dz v = \frac{1}{\rho} \nabla \times \int dz \vec{F} = \frac{1}{\rho} \nabla \times \vec{\tau}(z=0)$$

where we have again ignored spatial variations of $h,H$ , set the stress at the bottom to zero and ignored the horizontal components of the stress as well. As you will recall, this was the assumption inherent in the Sverdrup balance, that we could ignore bottom stress or the frictional term proportional to $r$. The above statement of the Sverdrup balance is more general that the previous one because we have explicitly included the surface wind stress and we have NOT made the assumption that the velocity $v$ is constant with depth. So the Sverdrup balance can be applied to the vertical integral of wind-driven currents (including the baroclinic velocity) and subject to some requirements on the 'levelness' of the ocean depth, is a useful description of wind-driven, frictionless (except for the surface layer) flow. Note that it does

include the wind driving and therefore the dynamics of the surface Ekman layer.

**Geostrophic and Ekman components of the Sverdrup Circulation**

The Sverdrup balance is a vertical integral of the simplified potential vorticity equation. Let's step back now and look at what velocities make up this balance. We will be using the more general equations for baroclinic flow here, which you will recall are the following:

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} - ru + \frac{F^x}{\rho},$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} - rv + \frac{F^y}{\rho}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

We could now insert the form of the body force derived above and we would have the full turbulent equations used in large-scale physical oceanography. But here we are interested in the Sverdrup balance, which is for a steady, linearized ocean, with wind forcing but otherwise no explicit friction. So the above become

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\tau_z{}^x}{\rho},$$

$$+fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\tau_z{}^y}{\rho}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

Suppose we write the velocity components as the sum of a geostrophic part and an Ekman part with superscripts denoting these. The first two equations become

$$-fv = -fv^g - fv^e = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau^x}{\partial z}$$

$$fu = -fu^g - fu^e = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau^y}{\partial z}$$

We have already looked at the geostrophic balance. Now we look further at the Ekman balance, integrating the Ekman balance between the free surface (z=0) and a depth at the base of the Ekman layer (z = -Z$_e$).

$$\int dz u^e = \frac{\tau^y(z=0)}{\rho f},$$

$$\int dz v^e = -\frac{\tau^x(z=0)}{\rho f},$$

$$\int dz\left[\frac{\partial u^e}{\partial x} + \frac{\partial v^e}{\partial y}\right] = w^e(z=-Z_e) = \frac{1}{\rho}\nabla_h \times \left[\frac{\vec{\tau}(z=0)}{f}\right]$$

In the first two equations above, we see that the integrated Ekman transport is to the right of the applied wind stress. In the third, we see that there is a

horizontal divergence of the net Ekman transport which will produce a vertical velocity at the bottom of the Ekman layer (the vertical velocity at the surface being zero). These expressions are not valid at the equator where the Corilios parameter vanishes! The vertical velocity is such that there is sinking with subtropical gyres and upwelling within subpolar gyres, irrespective of hemisphere.

Now if we know the integrated Ekman transport and we also know the vertically integrated Sverdrup transport, then the vertically integrated geostrophic transport must just be the difference between the Sverdrup and Ekman transports, since by definition, $v^s = v^g + v^e$. Some investigators have used this constraint to help determine the unknown reference level velocity for geostrophic motion, but this is only as good as the assumptions behind it. In fact, you will be examining this in one of your next homework problems. The fact that there are **large** difference between the actual and Sverdrup volume transports in the Gulf Stream (150 vs. 30 Sv., respectively) should make one approach this method with caution! We will be seeing this difference later.

**Munk's gyre vs. Stommel's gyre**

Munk (1950) produced another model of the N. Atlantic wind-driven circulation and explained its westward intensification using a different friction model than Stommel. Instead of vertical friction as contained in the frictional parameter, $r$ opposing the motion, Munk used a horizontal eddy viscosity. We will not go into this model, as it also required a positive $\beta$ to have the Gulf Stream stronger on the western boundary. We will just write down the linear vorticity balance with **both** frictional terms.

$$-A^h(\psi_{xxxx} + \psi_{yyyy}) + r(\psi_{xx} + \psi_{yy}) + \beta\psi_x = -\frac{1}{\rho}[\tau_x^y(z=0) - \tau_y^x(z=0)]$$

Here we have replaced the body force with the wind stress at the surface and the streamfunction is now the vertical integral of the previous streamfunction:

$$\int dz v = \psi_x \ \& \int dz u = -\psi_y$$

Munk solved the above neglecting the bottom friction term (proportional to $r$) but retaining horizontal friction (proportional to $A^h$)

Because there are now four x-derivitives to the streamfunction, the western boundary current is even sharper and more intense than Stommel's, which had 2 x-derivatives. As we indicated earlier, the interior Sverdrup Balance holds in the interior of the basin, but in the boundary near the western boundary, a different dynamical balance must be present. We can determine the width of the western boundary current in each of the two models very easily, without actually solving the problem. In each model we have the following approximate balance near the western boundary, where things change rapidly in the x-direction:

$$r\psi_{xx} + \beta\psi_x \approx 0 \rightarrow x_s \approx \left[\frac{r}{\beta}\right], Stommel's\ wbc\ width$$

$$A^h\psi_{xxxx} + \beta\psi_x \approx 0 \rightarrow x_m \approx \left[\frac{A^h}{\beta}\right]^{1/3}, Munk's\ wbc\ width$$

Time-dependent & non-linear numerical models of the ocean circulation using either bottom or lateral friction must be able to resolve the structure of the wbc. The above spatial scales for the two types of boundary currents become important minimum scales required for numerical ocean models.

This completes the extra notes on the wind-driven circulation. Further discussion will center around the textbook until we move on to other parts of the ocean.