Lecture 15

15.1 Administration

- Collect PS 7
- Distribute PS 8 (due November 13)

15.2 Curvature terms

(Q4). I brought this up last lecture, but I want to reiterate it: Note that the coriolis force is *not* aligned with the surface of the parabola (see figure 15.1).

It has a component tangent to the surface, $|(-2\mathbf{\Omega} \times \mathbf{u})| \cos\theta$ and a component normal to the surface $|(-2\mathbf{\Omega} \times \mathbf{u})| \sin\theta$. So the particle isn't turned to the right with the full force of the rotation (as it would be if the table were flat). Instead a portion goes into turning to the right and a portion goes into increasing the weight of the particle!

This becomes more obvious if we consider a rotating coordinate system that *isn't* coincident with the fixed coordinate system (see figure 15.2). In this *locally cartesian coordinate system* Ω has two components:

$$\Omega_z = \Omega \cos\phi \quad (15.1)$$

$$\Omega_y = -\Omega \sin\phi \quad (15.2)$$

So $\mathbf{\Omega}$ in this coordinate system is:

$$\mathbf{\Omega} = [0\mathbf{i} - \Omega \sin\phi \mathbf{j} + \Omega \cos\phi \mathbf{k}] \quad (15.3)$$

and

$$-2\mathbf{\Omega} \times \mathbf{u} = (v\Omega \cos\phi + w\Omega \sin\phi)\mathbf{i} \quad (15.4)$$

$$= -u\Omega \cos\phi \mathbf{j} - \mathbf{u}\Omega \sin\phi \mathbf{k} \quad (15.5)$$

You can see that if $\mathbf{u} = [u\ 0\ 0]$, then there is a Coriolis force in both the y and z directions.

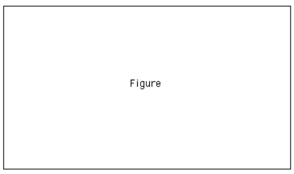

Figure 15.1: (`fig:Lec15Experiment1`) Coriolis force diagram. Note that the Coriolis force is not aligned with the surface of the parabola.

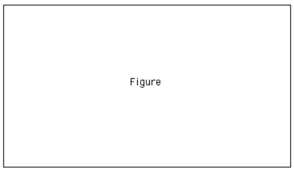

Figure 15.2: (`fig:Lec15Experiment2`) Same system as is seen in figure 15.1, but with a locally Cartesian coordinate system

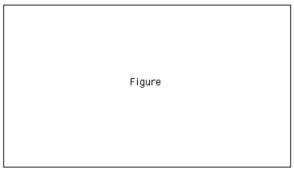

Figure 15.3: (`fig:Lec15Experiment3`) Same system as is seen in figure 15.1, but considering centrifugal force diagram with a locally Cartesian coordinate system.

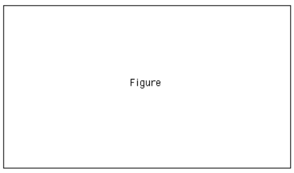

Figure 15.4: (`fig:Lec15Experiment4`) Same system as is seen in figure 15.1, but considering centrifugal forces due to the curvature of the parabola.

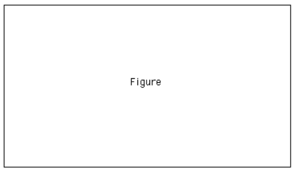

Figure 15.5: (`fig:Lec15LocalCartesian`) The locally Cartesian coordinate system on the sphere.

This is similar to when considering the centrifugal force (see figure 15.3). Note that u_θ motion will also generate centrifugal forces due to the curvature of the parabola as seen in figure 15.4. Here there is one component in the direction of coriolis, one in direction of gravity. When go to a locally cartesian coordinate system, you need to include these curvature terms, like coriolis they are a consequence of the coordinate system.

On the sphere, the locally Cartesian coordinate system is as seen in figure 15.5. The associated momentum equations on an oblate spheroid are:

$$x: \qquad \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}[F_x] + \frac{uv\sin\phi}{r\cos\phi} - \frac{uw}{r} - 2\Omega w\cos\phi + 2\Omega v\sin\phi \tag{15.6}$$

Here the term with F_x is friction and goes as $\mu\nabla^2\mathbf{u}+\frac{\mu}{3}\nabla(\nabla\cdot\mu)$. The next two terms are centrifugal, and the final two terms are the Coriolis bits. For y and z we have

$$y: \qquad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}[F_y] - \frac{u^2\sin\phi}{r\cos\phi} - \frac{vw}{r} - 2\Omega u\sin\phi \tag{15.7}$$

$$z: \qquad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \frac{1}{\rho}[F_z] + \frac{u^2}{r} + \frac{v^2}{r} + 2\Omega u\cos\phi \tag{15.8}$$

Here all the extra centrifugal terms are called "curvature terms".

Note that in the spherical coordinate system the impact of the Coriolis and extra centrifugal terms is to make things lighter! The funny $\frac{uw}{r} \neq \frac{vw}{r}$ terms are a response to this. They have the form of a centrifugal force, are easier understood in terms of conservation of angular momentum. w is "lifting" x and y momentum, and by conservation of angular momentum, there must be a corresponding retarding force.

These centrifugal terms are neglected in every case I'm aware of. Said more honestly, I've never heard of an application where these terms are kept. They all have the earth's radius in the denominator which tends to make the terms small. The ocean modelers have in EAPS did a run where they kept all these terms. They only made a difference of 1 part in 1000. From here out, the curvature terms will be neglected.

15.3 Vorticity equation in a rotating system

So long as we are here, we may as well write down the vorticity equation. The left hand side and the first three terms on the right hand side are as they were for the non-rotating case.

$$\frac{D\omega}{Dt} = \omega\cdot\nabla\mathbf{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nu\nabla^2\omega \tag{15.9}$$

But now we have to figure out what to do with the $\nabla\times(2\mathbf{\Omega}\times\mathbf{u})$ term. Recall:

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) \tag{15.10}$$

$$\Rightarrow \quad \nabla\times(2\mathbf{\Omega}\times\mathbf{u}) = (\mathbf{u}\cdot\nabla)2\mathbf{\Omega} - (2\mathbf{\Omega}\cdot\nabla)\mathbf{u} + 2\Omega(\nabla\cdot\mathbf{u}) - \mathbf{u}(\nabla\cdot 2\mathbf{\Omega}) \tag{15.11}$$

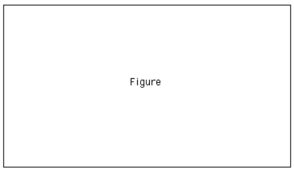

Figure 15.6: (`fig:Lec15VerticalSheer`) The shear tilt's a fluid line contains planetary vorticity into a new direction, adding vorticity in the x direction.

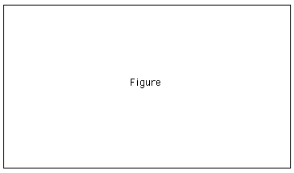

Figure 15.7: (`fig:Lec15Vortexstretching`) Change in vorticity through stretching and contracting.

If we are working with a compressible system, then

$$= \quad 2(\mathbf{u}\cdot\nabla)\mathbf{\Omega} - 2(\mathbf{\Omega}\cdot\nabla)\mathbf{u} - 2\mathbf{u}\nabla\cdot\mathbf{\Omega} \tag{15.12}$$

If we are in an Earth-centric coordinate system and $\mathbf{\Omega}$ is constant, then

$$= \quad -(2\mathbf{\Omega}\cdot\nabla)\mathbf{u} \tag{15.13}$$

Thus (again recalling that this is for a compressible fluid only),

$$\frac{D\omega}{Dt} = \omega\cdot\nabla\mathbf{u} - (-2\mathbf{\Omega}\cdot\nabla\mathbf{u}) + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nu\nabla^2\omega \tag{15.14}$$

$$\frac{D\omega}{Dt} = ((\omega + 2\mathbf{\Omega})\cdot\nabla)\mathbf{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nu\nabla^2\omega \tag{15.15}$$

We already know how the $\omega\cdot\nabla\mathbf{u}$ term creates changes in vorticity, that $\nabla\rho\times\nabla p$ creates vorticity through lines of force not passing through a blob's center of mass, and how vorticity is created by viscosity.

Now what about that $2\mathbf{\Omega}\cdot\nabla\mathbf{u}$ term? We can draw pictures just like those for $\omega\cdot\nabla\mathbf{u}$. See figure 15.6. Just like in $\omega\cdot\nabla\mathbf{u}$ you can change vorticity through stretching and contracting (conservation of angular momentum) (see figure 15.7).

Note that $\omega\cdot\nabla\mathbf{u}$ generated vorticity changes by stretching and tilting vortex lines. $2\Omega\cdot\nabla\mathbf{u}$ generates vorticity changes by stretching and tilting *fluid* lines. Planetary vorticity is always present in the fluid, just waiting to pop out and bite you. It is exactly for this reason that potential flow just doesn't exist on the planetary scale.

15.4 Kelvin's Theorem in a rotating system

Does $\frac{d\Gamma}{dt} = 0$ still hold in a rotating frame? Yes and no. Yes, Kelvin's Theorem is still valid but Γ takes on a new form. You might recall that when we proved Kelvin's theorem we took the dot product of each term in the momentum equation with $d\mathbf{x}$:

$$\frac{d\Gamma}{dt} = \int_c \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{x} + \oint_c \mathbf{u}\cdot\frac{D}{Dt}(d\mathbf{x}) \tag{15.16}$$

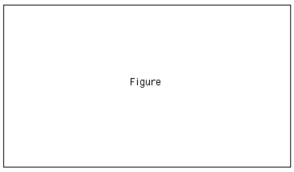

Figure 15.8: (`fig:Lec15Parellelogram`) Area of this parallelogram given by $|\mathbf{u}\Delta t \times dx|$, normal of the area is given by the $\mathbf{u}\Delta t \times d\mathbf{x}$ direction.

All the terms in the momentum equation for an inertial frame go to zero, but in the rotating frame we have the extra term:

$$\int_c (-2\mathbf{\Omega} \times \mathbf{u}) \cdot d\mathbf{x} \tag{15.17}$$

Using the vector identity

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} &= \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) & (15.18) \\ \Rightarrow \qquad \int_c (-2\mathbf{\Omega} \times \mathbf{u}) \cdot d\mathbf{x} &= \int_c -2\Omega \cdot (\mathbf{u} \times d\mathbf{x}) & (15.19) \end{aligned}$$

$\mathbf{u} \times d\mathbf{x}$ has a magnitude given by $|\mathbf{u}|\,|d\mathbf{x}|\,\sin\theta$, where θ is defined as the angle between them, and a direction perpendicular to both. What does this look like? See figure 15.8.

So, call $d\mathbf{A} = \mathbf{u}\Delta t \times d\mathbf{x} =$ the small area, then $\mathbf{u} \times d\mathbf{x} = \frac{d}{dt}(d\mathbf{A})$ in the limit. We can also write this as

$$\mathbf{u} \times d\mathbf{x} = \frac{D}{Dt}(d\mathbf{A}) \tag{15.20}$$

since we are following the flow. Substitution yields

$$\oint_c (-2\mathbf{\Omega} \times \mathbf{u}) \cdot d\mathbf{x} = \oint -2\mathbf{\Omega} \cdot (\mathbf{u} \times d\mathbf{x}) = \iint_A -2\mathbf{\Omega} \cdot \frac{D}{Dt}(d\mathbf{A}) \tag{15.21}$$

Can pull the $\frac{D}{Dt}$ outside the integral if you assume $\mathbf{\Omega}$ is constant as we do.

$$\oint_c (-2\mathbf{\Omega} \times \mathbf{u}) \cdot d\mathbf{x} = \frac{D}{Dt} \iint_A -2\mathbf{\Omega} \cdot d\mathbf{A} \tag{15.22}$$

Why is that useful? In the inertial frame we found

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint_c \mathbf{u} \cdot d\mathbf{x} = \frac{d}{dt} \iint_A \omega \cdot d\mathbf{A} = 0 \tag{15.23}$$

We just learned that

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \iint_A -2\mathbf{\Omega} \cdot d\mathbf{A} = \frac{d}{dt} \iint_A \omega \cdot d\mathbf{A} = 0 \tag{15.24}$$

Thus,

$$\begin{aligned} \frac{d}{dt} \iint_A \omega \cdot d\mathbf{A} + \frac{d}{dt} \iint_A 2\mathbf{\Omega} \cdot d\mathbf{A} &= 0 & (15.25) \\ \frac{d}{dt} \iint_A (\omega + 2\mathbf{\Omega}) \cdot d\mathbf{A} &= 0 & (15.26) \end{aligned}$$

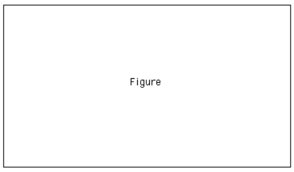

Figure 15.9: (`fig:Lec15fig:Lec15Scales`) The horizontal length scales are much larger than the vertical length scales.

Will probably write

$$\frac{D\Gamma}{Dt} = \iint\limits_A (\omega + 2\mathbf{\Omega}) \cdot d\mathbf{A} = 0 \tag{15.27}$$

Define Γ_a as the circulation due to absolute vorticity $= \omega + 2\mathbf{\Omega}$. Then,

$$\Gamma_a = \iint\limits_A (\omega + 2\mathbf{\Omega}) \cdot d\mathbf{A} \tag{15.28}$$

and

$$\frac{d\Gamma_a}{dt} = \frac{d}{dt} \iint\limits_A (\omega + 2\mathbf{\Omega}) \cdot d\mathbf{A} = 0 \tag{15.29}$$

This is Kelvin's circulation theorem in a rotating frame. We'll return to this later.

15.5 Simplifications to the momentum equations

We're getting to the good stuff, but first we need to slog through some simplifications to the momentum equations. Recall

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{1}{\rho}\nabla p - 2\mathbf{\Omega} \times \mathbf{u} + \upsilon\nabla^2\mathbf{u} + \frac{\upsilon}{3}\nabla(\nabla \cdot \mathbf{u}) \tag{15.30}$$

Assume incompressible

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{1}{\rho}\nabla p - 2\mathbf{\Omega} \times \mathbf{u} + \upsilon\nabla^2\mathbf{u} \tag{15.31}$$

Expand

$$x: \quad \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega w \cos\phi + 2\Omega v \sin\phi + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{15.32}$$

$$y: \quad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi + \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{15.33}$$

$$z: \quad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + \upsilon\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{15.34}$$

Now we'll try and kill some terms.

Before killing terms, a statement that can be made about large-scale geophysical flows is that they have horizontal scales that are much larger than vertical scales (see figure 15.9). Keeping that in mind, consider the velocity divergence under $\nabla \cdot \mathbf{u} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{15.35}$$

$$\text{scales}: \quad \frac{U}{L} \quad \frac{U}{L} \quad \frac{W}{H} \tag{15.36}$$

Imagine flow distributed more or less evenly between the three terms, then

$$\frac{W}{H} \cong \frac{U}{L} \Rightarrow W \cong \frac{H}{L}U \tag{15.37}$$

Since $L \gg H$, then $U \gg W$ for this to hold. Scaling in the x-direction

$$\begin{aligned}
x: \quad \frac{Du}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega w\cos\phi + 2\Omega v\sin\phi + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) && (15.38)\\
\frac{u^2}{L} \qquad & \qquad \frac{p}{\rho L} \qquad \Omega W \qquad \Omega U \qquad \nu\left[\frac{U}{L^2} \quad \frac{U}{L^2} \quad \frac{U}{H^2}\right] && (15.39)\\
x: \quad \frac{Du}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi + \nu\frac{\partial^2 u}{\partial z^2} && (15.40)
\end{aligned}$$

This last step was made noting that $L \gg H$, then $U \gg W$. Similar arguments for y direction

$$y: \quad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \nu\frac{\partial^2 v}{\partial z^2} \tag{15.41}$$

The z direction is trickier

$$\begin{aligned}
z: \quad \frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - g + 2\Omega\cos\phi + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] && (15.42)\\
\frac{w^2}{L} \qquad & \qquad \frac{p}{\rho H} \qquad g \qquad \Omega U \qquad \nu\left[\frac{w}{L^2} \quad \frac{w}{L^2} \quad \frac{w}{H^2}\right] && (15.43)
\end{aligned}$$

We can drop the first term by looking at the characteristic time-scales which go like

$$T \geq \frac{1}{\Omega} \tag{15.44}$$

so in this case

$$\frac{W^2}{L} \approx \frac{W}{T} \approx W\Omega \tag{15.45}$$

Since $U \gg W$, then $W\Omega \ll \Omega U \Rightarrow$. Thus we can kill $\frac{Dw}{Dt}$. Next, since $L \gg H$, $\frac{W}{L^2} \ll \frac{W}{H^2}$ and thus we can kill $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$. Finally, we have to make the argument that we can't have the remaining friction term dominate the coriolis term (else all z motion goes away). This motivates comparing the following two terms

$$\upsilon\frac{W}{H^2} \geq U\Omega \tag{15.46}$$

Since $W << U$ can kill $\nu\frac{W}{H^2}$ with respect to ΩU term. **(Jim's note to self – Do this better).**

So, we are down to three terms. How does that coriolis term compare with the pressure and gravity terms? Have to plug in numbers. Compare ΩU with g

$$\begin{aligned}
\Omega &= 7\times 10^{-5} && (15.47)\\
U &= 10.0\,\mathrm{m/s} \text{ or } 0.01\,\mathrm{m/s} && (15.48)\\
g &= 10 && (15.49)\\
& && (15.50)
\end{aligned}$$

$\Rightarrow \Omega U \ll g \Rightarrow$ kill coriolis term. That leaves the hydrostatic balance

$$z: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} - g \tag{15.51}$$

Put it all together, defining f = coriolis parameter $= 2\Omega \sin\phi$:

$$x: \quad \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \nu\frac{\partial^2 u}{\partial z^2} \tag{15.52}$$

$$y: \quad \frac{Dv}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + fu + \nu\frac{\partial^2 v}{\partial z^2} \tag{15.53}$$

$$z: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{15.54}$$

$$\tag{15.55}$$

Now for one of the dirty little secrets of GFD... eddy diffusivity. For incompressible N-S, the friction terms had the form

$$F_x = \upsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{15.56}$$

We just argued that the first two terms are small compared to the third because of the length scales in the denominator. **(Jim – This seems like a rather abrupt end).**

15.6 Reading

KC01: 14.5
CR94: 4.1-4.2, 3.4