Energy equation

The Lagrangian form of the energy equation is

$$\frac{dE}{dt} = \dot{Q} - \dot{W} \tag{1}$$

$$E \equiv \text{total energy}$$

$$\dot{Q} \equiv \text{rate of heat transfer; + when added to system}$$

$$\dot{W} \equiv \text{rate of work; + when done by system}$$

Here, E is our extensive property. The associated intensive property will be made up of two terms:

$$e \equiv \text{internal energy per unit mass}$$

$$\frac{1}{2}\mathbf{u}\cdot\mathbf{u} = \frac{1}{2}\mathbf{u}^2 \equiv \text{kinetic energy per unit mass}$$

Thus, the intensive property is $\rho(e + \frac{1}{2}\mathbf{u}^2) \equiv$ energy/volume, where $de = c_v dT, c_v \equiv$ specific heat at constant volume. c_v is the ratio of the amount of heat, Q, it takes to raise a mass, m, by an amount ΔT. $c_v = Q/m\Delta T$. RTT says that

$$\frac{dE}{dT} = \iiint_V \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) dV = \dot{Q} - \dot{W} \tag{2}$$

Recall this? When an intensive property is a product of ρ, the constant equation wipes out the $\rho(e + \frac{1}{2}\mathbf{u}^2)\nabla \cdot \mathbf{u}$ term.

Now consider $\dot{Q}$, the heat transfer bit. We want it in integral form, and it's the heat transfer across the boundaries of our blob, so it will be a surface integral.

$$\dot{Q} = -\iint_A \mathbf{q}\cdot d\mathbf{A} \tag{3}$$

What's up with the sign? $\dot{Q}$ is positive when work is added to the system, but something fluxing into a control volume has a negative sign since $d\mathbf{A}$ is positive outward.

Now for the $\dot{W}$ term. Work is force times distance, $W = \mathbf{F}\cdot\Delta\mathbf{x}$. So the work rate will be force times distance divided by time, $\dot{W} = \mathbf{F}\cdot\frac{\Delta\mathbf{x}}{\Delta t}$, or force times velocity

$$\dot{W} = \mathbf{F}\cdot\mathbf{u} \tag{4}$$

Recall from our derivation of the momentum equations, that the forces acting on our blob of fluid are body and surface forces:

$$\mathbf{F} = \mathbf{F}_B + \mathbf{F}_S \tag{5}$$

This ignores the line force, which is surface tension. Thus

$$\dot{W} = \mathbf{F}_B\cdot\mathbf{u} + \mathbf{F}_S\cdot\mathbf{u} \tag{6}$$

From our work on the momentum equations, the only body force that we are considering so far, is gravity. It acts on all the mass in the blob, so we have the volume integral

$$\mathbf{F}_B \cdot \mathbf{u} \quad = \quad -\iiint\limits_V \rho \mathbf{g} \cdot \mathbf{u} \, dV \tag{7}$$

Note that $\dot{W} > 0$ when work is done *by* the system, and since $\mathbf{g}$ is *on* the system, $\mathbf{F}_B \cdot \mathbf{u} < 0$. Hence the minus sign.

Again, from our work on momentum, we went through a lot of pain to convince ourselves that the surface force was given by the stress tensor. It's force on the surface of the blob, so it will be an area integral

$$\mathbf{F}_S \cdot \mathbf{u} \quad = \quad -\iint\limits_A (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} \tag{8}$$

$\dot{W} > 0$ when work is done *by* the system, but here we have work being done *on* the system, so we need the minus sign.

We now have all the terms for the energy equation (1):

$$\begin{aligned}
\frac{dE}{dt} &= \dot{Q} - \dot{W} \\
\iiint\limits_V \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) dV &= -\iint\limits_A \mathbf{q} \cdot d\mathbf{A} - \left[-\iiint\limits_V \rho \mathbf{g} \cdot \mathbf{u} \, dV - \iint\limits_A (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} \right] \\
\iiint\limits_V \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) dV &= \iiint\limits_V \rho \mathbf{g} \cdot \mathbf{u} \, dV + \iint\limits_A (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} - \iint\limits_A \mathbf{q} \cdot d\mathbf{A} \qquad (9)
\end{aligned}$$

Now we need to convert the two area integrals on the RHS into volume integrals using the divergence theorem:

$$\iint\limits_A (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} \quad = \quad \iiint\limits_V \nabla \cdot (\tau \cdot \mathbf{u}) dV \tag{10}$$

$$-\iint\limits_A \mathbf{q} \cdot d\mathbf{A} \quad = \quad -\iiint\limits_V \nabla \cdot \mathbf{q} \, dV \tag{11}$$

Inserting these results in the energy equation (9) yields

$$\begin{aligned}
\iiint\limits_V \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) dV &= \iiint\limits_V \rho \mathbf{g} \cdot \mathbf{u} \, dV + \iiint\limits_V \nabla \cdot (\tau \cdot \mathbf{u}) dV - \iiint\limits_V \nabla \cdot \mathbf{q} \, dV \\
\iiint\limits_V \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) dV &= \iiint\limits_V \left[\rho \mathbf{g} \cdot \mathbf{u} + \nabla \cdot (\tau \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}\right] dV \qquad (12)
\end{aligned}$$

Since this integral must hold for all volumes

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) \quad = \quad \rho \mathbf{g} \cdot \mathbf{u} + \nabla \cdot (\tau \cdot \mathbf{u}) - \nabla \cdot \mathbf{q} \tag{13}$$

This is the energy equation. Note rising motion leads to an increase in potential energy as $\mathbf{g} \cdot \mathbf{u} < 0$, as it should since it is on the RHS. The terms:

$$\begin{aligned}
\rho \mathbf{g} \cdot \mathbf{u} &\equiv \text{changes in potential energy} \\
-\nabla \cdot \mathbf{q} &\equiv \text{changes due to heat flux} \\
\nabla \cdot (\tau \cdot \mathbf{u}) &\equiv \text{has a kinetic energy part and an internal heating part}
\end{aligned}$$

Let's look into the term containing stress a bit more deeply as it is clearly the most painful of the terms:

$$\nabla \cdot (\tau \cdot \mathbf{u}) = \mathbf{u} \cdot (\nabla \cdot \tau) + \tau : \nabla \mathbf{u} \tag{14}$$

The first term, $\mathbf{u} \cdot (\nabla \cdot \tau)$, is the rate of work done by the surface forces. Imbalances in stress accelerate the fluid blob and change its kinetic energy. The second term, $\tau : \nabla \mathbf{u}$, is the work done by deformation. Not all stress moves the blob – some of it deforms and causes heat change. Note the double contraction, $\mathbf{A} : \mathbf{B} \Rightarrow$ scalar.

The $\mathbf{u} \cdot (\nabla \cdot \tau)$ terms is the easiest of the two since we came up with an expression for $\nabla \cdot \tau$ when we looked at momentum conservation.

$$\begin{aligned} \nabla \cdot \tau &= \nabla \cdot \left(2\mu \mathbf{e} - p - \frac{2}{3}\mu \nabla \cdot \mathbf{u} \right) \\ &= -\nabla p + \mu \nabla^2 \mathbf{u} + \frac{\mu}{3} \nabla (\nabla \cdot \mathbf{u}) \\ \Rightarrow \quad \mathbf{u} \cdot (\nabla \cdot \tau) &= \mathbf{u} \cdot \left(-\nabla p + \mu \nabla^2 \mathbf{u} + \frac{\mu}{3} \nabla (\nabla \cdot \mathbf{u}) \right) \end{aligned} \tag{15}$$

We're advecting τ through our fixed blob of fluid. It is bringing in kinetic energy associated with the stress on the blob.

Now the second term,

$$\tau : \nabla \mathbf{u} \qquad \text{double contraction}$$

Notice, there is no $\cdot$ between the ∇ and the $\mathbf{u}$. $\nabla \mathbf{u}$ is the gradient of the velocity... The velocity gradient tensor!

$$\nabla \mathbf{u} = \mathbf{G} \tag{16}$$

Thus, this term looks like

$$\tau : \mathbf{G} \tag{17}$$

But $\mathbf{G} = \mathbf{e} + \frac{1}{2}\mathbf{r}$,

$$\tau : \left(\mathbf{e} + \frac{1}{2}\mathbf{r} \right) \tag{18}$$

Recall that $\mathbf{r}$ is an antisymmetric tensor, and τ is a symmetric tensor. The double dot product of asymmetric and antisymmetric matrix is zero

$$\mathbf{A} : \mathbf{B} = \sum_i \sum_j A_{ij} B_{ij} \tag{19}$$

In 2D

$$\begin{aligned} \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} : \begin{bmatrix} 0 & -\frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 \end{bmatrix} &= \tau_{11}(0) + \tau_{12}\left(\frac{\omega}{2}\right) + \tau_{21}\left(-\frac{\omega}{2}\right) + \tau_{22}(0) \\ &= \tau_{12}\frac{\omega}{2} - \tau_{21}\frac{\omega}{2} \\ &= 0 \end{aligned} \tag{20}$$

This last line is due to the fact that since τ is symmetric, $\tau_{12} = \tau_{21}$. Thus, you can get rid of the $\frac{1}{2}r$ term and end up with

$$\tau : \mathbf{e} \tag{21}$$

Substitute in our expression for τ,

$$\left(2\mu\mathbf{e} - \left[p + \frac{2}{3}\mu\nabla\cdot\mathbf{u}\right]\mathbf{I}\right) : \mathbf{e}$$

$$2\mu\mathbf{e} : \mathbf{e} - p\mathbf{I} : \mathbf{e} - \frac{2}{3}\mu\nabla\cdot\mathbf{u}\mathbf{I} : \mathbf{e} \tag{22}$$

Let's see what (if anything) we can learn by performing these double contractions. We'll start with an easy one

$$\begin{aligned} p\mathbf{I} : \mathbf{e} &= p\frac{\partial u}{\partial x} + p\frac{\partial v}{\partial y} + p\frac{\partial w}{\partial z} \\ &= p\nabla\cdot\mathbf{u} \end{aligned} \tag{23}$$

That is, change in energy due to compression or expansion. The last term is also pretty easy

$$\begin{aligned} \frac{2}{3}\mu\nabla\cdot\mathbf{u}\mathbf{I} : \mathbf{e} &= \frac{2}{3}\mu\nabla\cdot\mathbf{u}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ &= \frac{2}{3}\mu\nabla\cdot\mathbf{u}(\nabla\cdot\mathbf{u}) \\ &= \frac{2}{3}\mu(\nabla\cdot\mathbf{u})^2 \end{aligned} \tag{24}$$

Finally, the first term $2\mu\mathbf{e} : \mathbf{e}$

$$\begin{aligned} \mathbf{e} : \mathbf{e} &= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} : \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \\ &= \begin{array}{l} e_{11}e_{11} + e_{12}e_{21} + e_{13}e_{31} + \\ e_{21}e_{12} + e_{22}e_{22} + e_{23}e_{32} + \\ e_{31}e_{13} + e_{32}e_{23} + e_{33}e_{33} \end{array} \end{aligned} \tag{25}$$

Note

$$e_{11}e_{11} = \left(\frac{\partial u}{\partial x}\right)^2 \tag{26}$$

$$\begin{aligned} e_{12}e_{21} &= \left[\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]^2 \\ &= \frac{1}{4}\left(\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2\right) \\ &= \frac{1}{4}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2 \end{aligned} \tag{27}$$

etc.

Consider just the $e_{11}e_{11}$ term. Remember,

$$2\mu\mathbf{e} : \mathbf{e} \quad \Rightarrow \quad 2\mu\left(\frac{\partial u}{\partial y}\right)^2 = 2\left(\mu\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y} \tag{28}$$

The $\mu\frac{\partial u}{\partial y}$ bit is associated with normal stress, and $\frac{\partial u}{\partial y}$ with the velocity gradient. Thus, this term has something to do with the work it takes to pull apart molecules in a blob of fluid. The other terms, such as $(\frac{\partial u}{\partial y})^2$ and $\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$, say something similar about shear and twisting. Clearly this is a bit of a mess. Thankfully, what is typically done is to sweep it all under the rug and define

$$\begin{aligned} \phi &= \left(2\mu \mathbf{e} : \mathbf{e} - \frac{2}{3}\mu(\nabla \cdot \mathbf{u})^2\right) \\ &\equiv \text{viscous component of the deformation work rate} \end{aligned} \tag{29}$$

This allows us to write

$$\tau : \nabla \mathbf{u} = -p(\nabla \cdot \mathbf{u}) + \phi \tag{30}$$

Tracking this all the way back to the energy equation gives:

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) = \rho \mathbf{g} \cdot \mathbf{u} + \mathbf{u} \cdot \left[-\nabla p + \mu \nabla^2 \mathbf{u} + \frac{\mu}{3}\nabla(\nabla \cdot \mathbf{u})\right] - p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \tag{31}$$

0.1 Thermal energy (or heat) equation

What a mess. What games can we play to make this more comprehensible? Let's go back to a version of the momentum equation:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} + \mathbf{u} \cdot (\nabla \cdot \tau) \tag{32}$$

Multiply (dot) through by $\mathbf{u}$

$$\rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) = \rho \mathbf{u} \cdot \mathbf{g} + \mathbf{u} \cdot (\nabla \cdot \tau) \tag{33}$$

This looks a little familiar. It is the mechanical energy equation. Subtract this from the total energy equation

$$\begin{aligned} \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) &= \rho \mathbf{g} \cdot \mathbf{u} + \mathbf{u} \cdot (\nabla \cdot \tau) - p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \\ - \quad \rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) &= \rho \mathbf{u} \cdot \mathbf{g} + \mathbf{u} \cdot (\nabla \cdot \tau) \end{aligned}$$

and we get the thermal energy (or heat) equation:

$$\Rightarrow \quad \rho \frac{De}{Dt} = -p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \tag{34}$$

$$\text{OR} \quad \rho C_v \frac{DT}{Dt} = -p(\nabla \cdot \mathbf{u}) + \phi - \nabla \cdot \mathbf{q} \tag{35}$$

The second equation is obtained using $de = C_v dT$. Here

$$\begin{aligned} -p(\nabla \cdot \mathbf{u}) &\equiv \text{change in thermal energy due to compression} \\ \phi &\equiv \text{change in thermal energy due to viscosity} \\ -\nabla \cdot \mathbf{q} &\equiv \text{change in thermal energy due to heat transfer} \end{aligned}$$

0.2 Approximations

Almost always it is safe to say that $\phi \ll 1$, and it is ignored (this broad statement should make you grumpy, but I'm not going to do the scaling here):

$$\rho C_v \frac{DT}{Dt} + p(\nabla \cdot \mathbf{u}) = -\nabla \cdot \mathbf{q} \tag{36}$$

Here, $\rho C_v \frac{DT}{Dt}$ is temperature changes at constant volume, and $p(\nabla \cdot \mathbf{u})$ is temperature changes due to changing volume.

Remember the Boussinesq approximation? I stated it as

$$\rho = \rho_o + \rho'(x, y, z, t) \tag{37}$$

And it allowed us to simplify the continuity equation to

$$\nabla \cdot \mathbf{u} = 0 \tag{38}$$

It would be tempting to simply remove the $-p(\nabla \cdot \mathbf{u})$ term, leaving

$$\rho C_v \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} \tag{39}$$

But this isn't correct, because here $\nabla \cdot \mathbf{u}$ is similar in scale to $\rho C_v \frac{DT}{Dt}$, so we'll keep it. If we can't say $\nabla \cdot \mathbf{u} = 0$, let's substitute in the continuity equation instead

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \qquad \Rightarrow \qquad \nabla \cdot \mathbf{u} = -\frac{1}{\rho}\frac{D\rho}{Dt} \tag{40}$$

Substitution yields

$$\rho C_v \frac{DT}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{q} \tag{41}$$

Rewrite as

$$\rho C_v \frac{DT}{Dt} + \rho p \left(-\frac{1}{\rho^2}\frac{D\rho}{Dt}\right) = -\nabla \cdot \mathbf{q} \tag{42}$$

Using the fact that $-\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{D}{Dt}\frac{1}{\rho}$

$$\rho C_v \frac{DT}{Dt} + \rho p \frac{D\alpha}{Dt} = -\nabla \cdot \mathbf{q} \tag{43}$$

where $\alpha = \frac{1}{\rho} \equiv$ specific volume.

To go the next step we need to bring in an equation of state. Traditional, linear equations of state (eg for the ocean) will work, but in the interest of familiarity and clarity I'll go with the equation of state for a perfect gas.

$$p = \rho RT \qquad \text{or} \qquad p\alpha = RT \tag{44}$$

Differentiate and rearrange:

$$\begin{aligned} p\frac{D\alpha}{Dt} + \alpha\frac{Dp}{Dt} &= R\frac{DT}{Dt} \\ \Rightarrow \qquad p\frac{D\alpha}{Dt} &= R\frac{DT}{Dt} - \alpha\frac{Dp}{Dt} \qquad (45) \\ & \qquad (46) \end{aligned}$$

Substitute:

$$\begin{aligned} \rho C_v \frac{DT}{Dt} + \rho R\frac{DT}{Dt} - \rho\alpha\frac{Dp}{Dt} &= -\nabla \cdot \mathbf{q} \\ \rho(C_v + R)\frac{DT}{Dt} - \frac{Dp}{Dt} &= -\nabla \cdot \mathbf{q} \\ \rho C_p \frac{DT}{Dt} - \frac{Dp}{Dt} &= -\nabla \cdot \mathbf{q} \qquad (47) \end{aligned}$$

Where $C_v + R = C_p$. Here, $\rho C_p \frac{DT}{Dt}$ is temperature change from heating at constant pressure, and $\frac{Dp}{Dt}$ is the correction term for constant pressure. Thus the approximations made are ϕ small and an ideal gas (though structurally similar to case with any equation of state).

The Boussinesq approximation says that $\frac{Dp}{Dt}$ is small. Hence

$$\rho C_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} \tag{48}$$

Further, using Fourier's Law of conduction

$$\mathbf{q} = -k \nabla T \tag{49}$$

which comes from measuring the heat flow through a block with different temperatures on each side, we get the following result

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot (k \nabla T) \tag{50}$$

Here $k \equiv$ thermal conductivity, and if constant we have

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T \tag{51}$$