Chapter 1

Shallow water gravity waves

1.1 Surface motions on shallow water

Consider two-dimensional (x-z) motions on a nonrotating, shallow body of water, of uniform density ρ, as shown in Fig. 1.1 below.

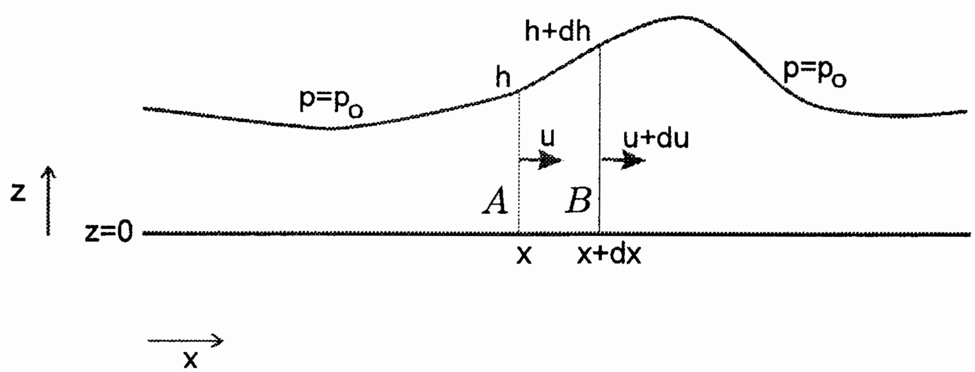

Figure 1.1: The shallow water system.

The flow is assumed to be inviscid and independent of the spatial dimension y (into the paper). We shall *assume* that the water is so shallow that the flow velocity $u(x,t)$ is constant with depth. (We'll see later under what conditions this is reasonable; for now, let's just assume it to be true.) At the free surface, located at height $z = h(x,t)$, pressure is equal to atmospheric pressure p_0, assumed constant and uniform.

Consider the volume of water bounded by the vertical surfaces A and B in the figure. These surfaces are located at x and $x + dx$ respectively. The mass of this volume, per unit length in y, is just $dm = \rho h\, dx$. Now, mass cannot be created or destroyed within the volume, so the only way it can change is because of the **fluxes** of mass across the interfaces A and B. Consider Fig. 1.2.

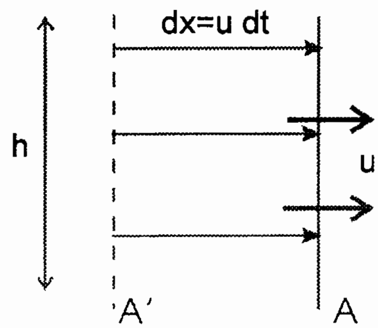

Figure 1.2: Illustrating the flux of mass across the interface A.

Since the velocity at A is u, in a time interval dt all the fluid between A' and A passes across A, where the distance between A' and A is $dx = u\ dt$. Thus, the area (i.e., the volume per unit length in y) passing across A in this time is $hu\ dt$, and so the mass (per unit length in y) is $\rho hu\ dt$. Therefore the **mass flux**—the mass crossing A per unit time, per unit length in y—is $\rho u(x)h(x)$. The mass flux across interface B is $\rho u(x + dx)h(x + dx)$ (directed toward positive x, out of the volume). Therefore the rate of accumulation of mass (per unit length in y) within the volume defined by AB is

$$\begin{aligned}\frac{\partial m}{\partial t} &= \rho u(x)h(x) - \rho u(x + dx)h(x + dx) \\ &= -\rho \frac{\partial (uh)}{\partial x}\, dx \;.\end{aligned}$$

Since $m = \rho h\, dx$, the factors of $\rho\, dx$ cancel, leaving us with

$$\frac{\partial h}{\partial t} = -\frac{\partial (uh)}{\partial x} \;.$$

Differentiating the RHS by parts and rearranging, we arrive at the **equation of continuity**:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = -h\frac{\partial u}{\partial x} \;. \tag{1.1}$$

This equation expresses the local rate of change of surface height in terms of two contributions:

(i) by advection of height $-u\partial h/\partial x$

(ii) by **volume convergence** $-h\partial u/\partial x$.

These two effects are depicted (both in a sense to increase h locally) in Fig. 1.3.

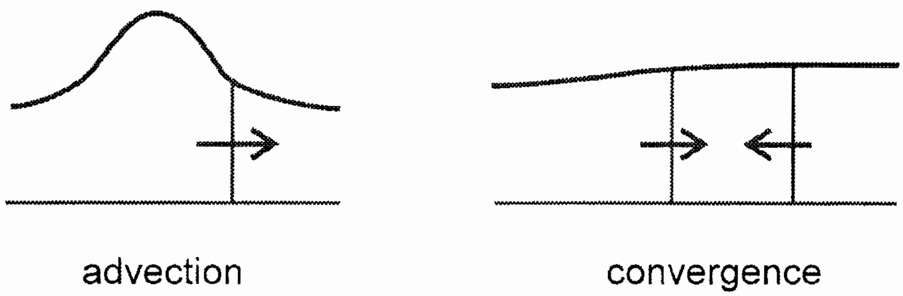

Figure 1.3: Contributions to $\partial h/\partial t$.

Now, in a similar way, consider the momentum balance of the water in the volume. We shall need to know the distribution of pressure p within the water. To do this, we use the principle of **hydrostatic balance**, which states that the pressure increases with depth according to the overhead mass per unit area. Specifically (see Fig. 1.1), the pressure at any depth $h - z$ below the surface is related to surface pressure by

$$p(z,t) = p_0 + \int_z^h \rho g \, dz = p_0 + \rho g(h - z) , \tag{1.2}$$

where g is the acceleration due to gravity (and both ρ and g are constants). The second term on the RHS of (1.2) simply represents the mass of water per unit area above level z. Newton's law of motion applied to the volume gives

$$m\frac{du}{dt} = F ,$$

where F is the net force (per unit length in y) applied to the volume. Since we are assuming friction to be negligible, the only such forces acting are pressure forces, which are as depicted in Fig. 1.4[1].

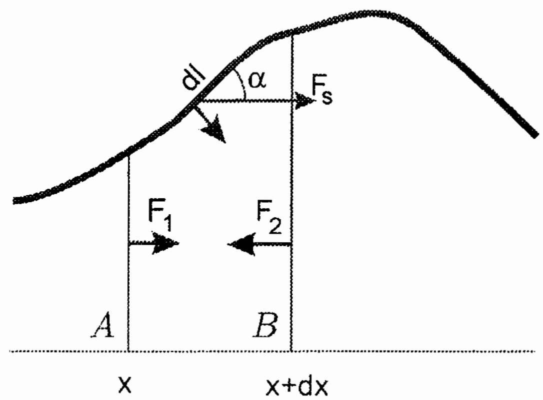

Figure 1.4: Forces acting on the fluid volume.

That acting on the volume across interface A (tending to accelerate the volume in the positive x direction) is equal to a force, per unit length in y, of $F_1 = \int_0^h p(x,z)\,dz$; that acting across interface B (tending to accelerate the volume in the negative x direction) is $F_2 = \int_0^h p(x+dx,z)\,dz$. However, there is a third component of the net force acting on the free surface, represented in the figure as F_s. Atmospheric pressure exerts a force $p_0 dl$ per unit length in y, where dl is the volume's width along the surface. Because the surface is tilted, this has a nonzero component $p_0 dl \sin\alpha$ acting in the positive x-direction, where α is the angle of the interface. Since $dl = dx/\cos\alpha$, this contribution to the x-force is

$$F_s = p_0 \frac{\partial h}{\partial x} dx$$

(since $\tan\alpha = \partial h/\partial x$). Therefore the net force on the volume, per unit length in y, is

$$F = p_0 \frac{\partial h}{\partial x} dx + \int_0^h p(x,z)\,dz - \int_0^h p(x+dx,z)\,dz \ .$$

[1]We are in fact neglecting here one contribution to the force felt at the surface, that due to surface tension. Surface tension effects are negligible for motions of large horizontal scale (typically a few cm.), so this analysis is restricted to these large scales. Small-scale motions for which surface tension effects are important are known as *capillary waves.*

But, from (1.2), we have

$$\begin{aligned}\int_0^h p\,dz &= \int_0^h p_0\,dz + \rho g \int_0^h (h-z)\,dz\,,\\ &= p_0 h + \frac{1}{2}\rho g h^2\,.\end{aligned}$$

So

$$\begin{aligned}\int_0^h p(x,z)\,dz - \int_0^h p(x+dx,z)\,dz &= p_0 h(x) - p_0 h(x+dx)\\ &\quad + \frac{1}{2}\rho g h^2(x) - \frac{1}{2}\rho g h^2(x+dx)\\ &= -p_0\frac{\partial h}{\partial x}dx - \rho g h\frac{\partial h}{\partial x}dx\,.\end{aligned}$$

Therefore the acceleration of the volume is given by

$$m\frac{du}{dt} = F = -\rho g h\frac{\partial h}{\partial x}dx\,.$$

Note that this is independent of surface pressure p_0 (the terms involving it have cancelled): the net force on the volume is entirely due to the pressure gradients within the water which, because of hydrostatic balance, are entirely due to gradients in surface height. Now, using our expression $m = \rho h\,dx$, the preceding equation gives us (cancelling the factors $\rho h\,dx$)

$$\frac{du}{dt} = -g\frac{\partial h}{\partial x}$$

Here the derivative d/dt is the *material derivative*—this tells us how the velocity of the marked volume changes *as it moves around*. We need to convert this into a form that tells us how u changes in fixed coordinates. Since $u = u(x,t) \equiv dx/dt$, we simply apply the chain rule to write

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

and thus to write our **equation of motion** in final form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}\,. \tag{1.3}$$

Like (1.1), this links the local rate of change of velocity to two terms:

(i) the pressure gradient term and

(ii) the *advection of momentum.*

The two equations (1.1) and (1.3) give us two *predictive* equations in the two unknowns $u(x,t)$ and $h(x,t)$, and so in principle tell us all we need to know to determine how this system will evolve, given initial and boundary conditions. The equations are *nonlinear* (through the advective terms) and have complex properties in general, but they become quite simple under circumstances where they can be linearized.

1.2 Small-amplitude shallow-water surface waves

Suppose now our shallow water system is motionless ($u = 0$), with uniform depth D; this state trivially satisfies eqs. (1.1) and (1.3). Now suppose we perturb this state, such that $u(x,t) = u'(x,t)$ and $h(x,t) = D + h'(x,t)$, where the perturbation is small in the sense that

(i) $|h'| \ll D$, and

(ii) $|u'| \ll L/T$,

where L and T are respectively length and time scales for the motion. Now, eq. (1.1) becomes

$$\frac{\partial h'}{\partial t} + u'\frac{\partial h'}{\partial x} = -(D + h')\frac{\partial u'}{\partial x} ,$$

since the derivatives of D are zero. We now replace $(D + h')$ by D (using assumption (i)) and neglect the second term compared to the first (since $\partial h'/\partial t \sim |h'|/T$ and $u'\partial h'/\partial x \sim |u'|\,|h'|/L$, so the ratio of the latter to the former is $|u'|/LT$, which is small by assumption (ii)), leaving the **linearized** equation

$$\frac{\partial h'}{\partial t} = -D\frac{\partial u'}{\partial x} . \tag{1.4}$$

Similarly, (1.3) becomes

$$\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x} = -g\frac{\partial h'}{\partial x} ;$$

again we can use assumption (ii) to neglect the second term, leaving

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} \,. \tag{1.5}$$

We can now combine the two equations (1.4) and (1.5) to get a single equation for h, by combining $\partial/\partial t$ of (1.4):

$$\frac{\partial^2 h'}{\partial t^2} = -D\frac{\partial^2 u'}{\partial x \partial t}$$

with $\partial/\partial x$ of (1.5):

$$\frac{\partial^2 u'}{\partial x \partial t} = -g\frac{\partial^2 h'}{\partial x^2}$$

to give

$$\frac{\partial^2 h'}{\partial t^2} - gD\frac{\partial^2 h'}{\partial x^2} = 0 \,. \tag{1.6}$$

This is a **wave equation**, which describes how small-amplitude surface height perturbations evolve.

1.3 Background theory—nondispersive waves

1.3.1 Oscillations

Oscillations (*e.g.*, small amplitude oscillations of a simple pendulum) are often described by an equation of the form

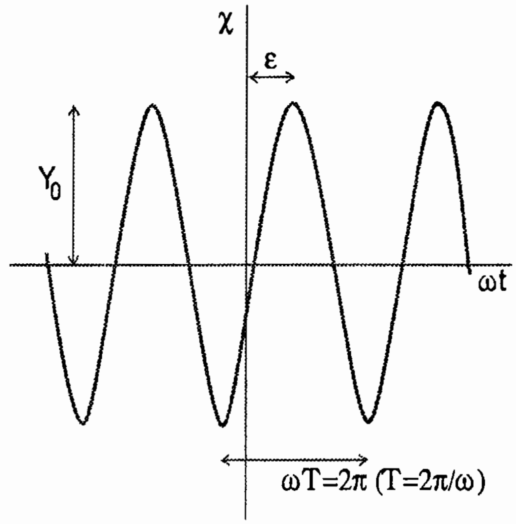

Figure 1.5: Characteristics of a simple oscillation.

$$\frac{d^2\chi}{dt^2} + \Omega^2\chi = 0 \tag{1.7}$$

where t is time and χ is some system variable (angular displacement in the case of the simple pendulum). (1.7) has solutions of the form

$$\chi(t) = Re\left(Y\, e^{i\omega t}\right) = Y_0 \cos(\omega t - \epsilon) \tag{1.8}$$

where frequency $\omega = \pm\Omega$, amplitude Y_0 and phase ϵ are real constants, and $Y = Y_0 e^{-i\epsilon}$ is a complex amplitude. Thus, as shown in Fig. 1.5, an oscillation is characterized by three constants: amplitude, frequency, and phase.

1.3.2 Nondispersive waves

Unlike such simple oscillations, waves are functions of both time and space. The simplest wave equation, of which (1.6) is an example, is of the form

$$\frac{\partial^2\chi}{\partial t^2} - c_0^2\frac{\partial^2\chi}{\partial x^2} = 0 \tag{1.9}$$

where c_0 is some constant. (In our case, χ represents surface height perturbations on shallow water of depth D and $c_0 = \sqrt{gD}$. However, it could equally well represent the electric or magnetic field *in vacuo*, with c_0 the speed of light; or pressure perturbations in a compressible fluid, with c_0 the sound speed.)

We can find solutions to (1.9) by *separating the variables*, writing

$$\chi(x,t) = Re\,[A(t)B(x)] \ .$$

More specifically, if we look for "wave-like" solutions for which $B(x) = e^{ikx}$, where k is a real wavenumber (so $2\pi/k$ is wavelength)[2], then $d^2B/dx^2 = -k^2B$, so

$$\frac{\partial^2\chi}{\partial x^2} = Re\left[A(t)\frac{d^2B}{dx^2}(x)\right] = -k^2 Re\,[A(t)B(x)] \ ,$$

and (1.9) becomes

$$\frac{d^2A}{dt^2} + k^2c_0^2A = 0 \ .$$

This has solutions like

$$A = \chi_+ e^{-i\omega t} \ ; \ A = \chi_- e^{+i\omega t}$$

where χ_+ and χ_- are constant (complex) amplitudes and the frequency ω satisfies

$$\omega^2 = k^2c_0^2 \ . \tag{1.10}$$

The full solution is

$$\chi(x,t) = Re\left[\chi_+ e^{i(kx-\omega t)} + \chi_- e^{i(kx+\omega t)}\right] \ . \tag{1.11}$$

Each of the two terms in (1.11) describes a **progressive wave** (Fig. 1.6):

[2]In general, any function of x can be expressed as a Fourier integral of such waves.

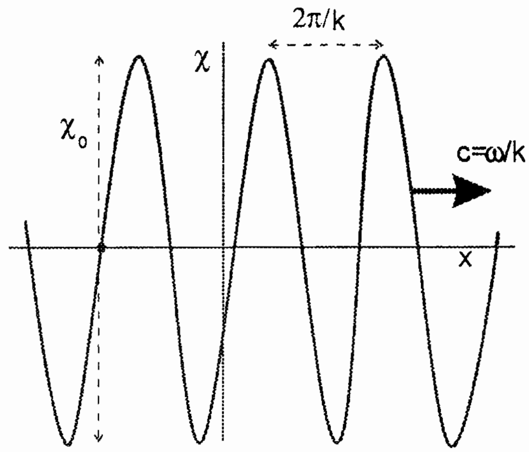

Figure 1.6: Characteristics of a progressive wave.

- at any instant, it is just a sinusoidal wave disturbance, of **wavelength** $2\pi/k$

- at any fixed location $x = x_0$, it is just an oscillation of the form $De^{\pm i\omega t}$, where $D = \chi_{\pm}e^{ikx_0}$ is its complex amplitude, of **period** $T = 2\pi/\omega$

- it **propagates** with **phase speed** $c = \omega/k = \pm c_0$. Note from (1.11) that χ is constant along characteristics with $kx \pm \omega t$ =constant, *i.e.*, $x = \mp\frac{\omega}{k}t$+constant.

Eq. (1.10) is the **dispersion relation** for the wave: for a given wavenumber k, it tells us the wave's frequency. This form is particularly simple, as shown in Fig. 1.7.

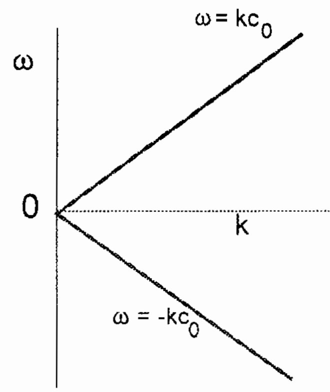

Figure 1.7: The dispersion relation for 1-D shallow water waves.

[Note that this is drawn for positive k only—we may define k positive, without loss of generality, as long as we do not try to constrain the sign of ω.] The phase speed $c = \omega/k = \pm c_0$; the waves can propagate in either direction.

These waves are **nondispersive**, *i.e.*, their phase speed is independent of wavenumber. Thus, all waves, of any wavenumber, propagate at the same speed (in either direction), which means that non-sinusoidal disturbances propagate *without change of shape.* In fact, any function

$$\chi(x,t) = F(x \pm c_0 t) \tag{1.12}$$

is a solution to (1.9)[3]. Eq. (1.12) just describes any shape of disturbance, including a localized one, that propagates at speed c without changing its shape (Fig. 1.8).

[3]To see this, note that if $X = x \pm c_0 t$, then $F = F(X)$ and the chain rule gives us

$$\begin{aligned}
\frac{\partial \chi}{\partial x} &= \frac{\partial F}{\partial x} = \frac{dF}{dX}\frac{\partial X}{\partial x} = \frac{dF}{dX}\,; \\
\frac{\partial^2 \chi}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{dF}{dX}\right) = \frac{\partial X}{\partial x}\frac{d^2F}{dX^2} = \frac{d^2F}{dX^2}\,; \\
\frac{\partial \chi}{\partial t} &= \frac{\partial F}{\partial t} = \frac{dF}{dX}\frac{\partial X}{\partial t} = \pm c_0\frac{dF}{dX}\,; \\
\frac{\partial^2 \chi}{\partial t^2} &= \frac{\partial}{\partial t}\left(\pm c_0\frac{dF}{dX}\right) = \pm c_0\frac{\partial X}{\partial t}\frac{d^2F}{dX^2} = c_0^2\frac{d^2F}{dX^2}\,.
\end{aligned}$$

So, (1.9) is satisfied by (1.12).

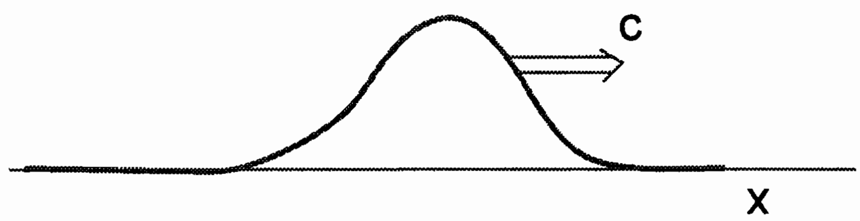

Figure 1.8: Nondispersive waves: arbitrary disturbances propagate without change of shape.

1.3.3 Two-dimensional waves

In two dimensions (x, y), (1.9) is replaced by

$$\frac{\partial^2 \chi}{\partial t^2} - c_0^2 \nabla^2 \chi = 0 \tag{1.13}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We can now look for **plane wave** solutions of the form

$$\chi(x, y, t) = Re\left[A_0 \, e^{i(kx+ly-\omega t)}\right] . \tag{1.14}$$

Then

$$\nabla^2 \chi = -\kappa^2 Re\left[A_0 \, e^{i(kx+ly-\omega t)}\right]$$

where $\kappa = \sqrt{k^2 + l^2}$ is the total wavenumber. Therefore, substituting into (1.13) gives the dispersion relation for this case

$$\omega^2 = \kappa^2 c_0^2 . \tag{1.15}$$

[Note that the one-dimensional case we discussed above is just a special case of the two-dimensional problem, with $l = 0$.]

Eq. (1.14) describes a plane wave because χ is constant along lines of constant phase $kx + ly - \omega t = constant$, so at any instant in time, $kx + ly =$ constant; see Fig. 1.9.

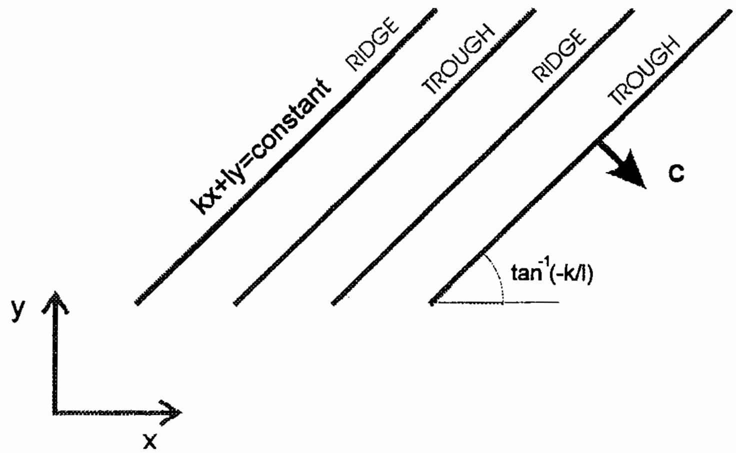

Figure 1.9:

The wave pattern moves at right angles to the phase lines, with speed c.

Plane waves are a special, and particularly simple, form of 2-D waves. Exactly what shape the wavefronts have will in general depend on the geometry of the system and of the process that generated the wave. If the source is very localized (*e.g.*, a stone dropped into water), the wavefronts will be circular, as shown in Fig. 1.10. Note that, far from the source (in the dashed rectangle), the wavefronts will look almost plane.

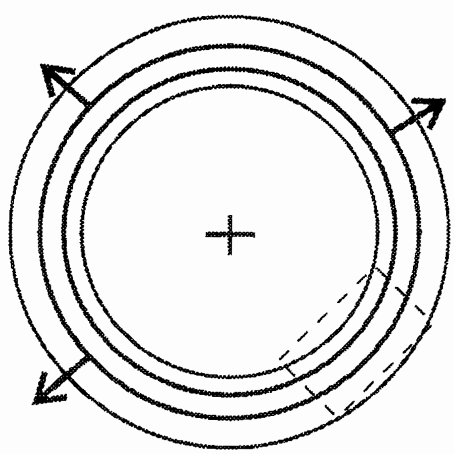

Figure 1.10: Circular wave fronts radiating from a localized source.

1.4 Motions within a wave

Returning to the 1-D problem, we can write the general solution, for a given wavenumber k, to (1.6) as

$$h'(x,t) = Re\left[H_+ e^{ik(x-c_0t)} + H_- e^{ik(x+c_0t)}\right] , \tag{1.16}$$

the first term representing a sinusoidal wave propagating to the right, the second one propagating to the left. Now, from (1.5),

$$u'(x,t) = Re\left[\frac{c_0}{D}H_+ e^{ik(x-c_0t)} - \frac{c_0}{D}H_- e^{ik(x+c_0t)}\right] . \tag{1.17}$$

So, for that mode propagating to the right (left), velocity and height perturbations are in phase (in antiphase), as shown in Fig. 1.11.

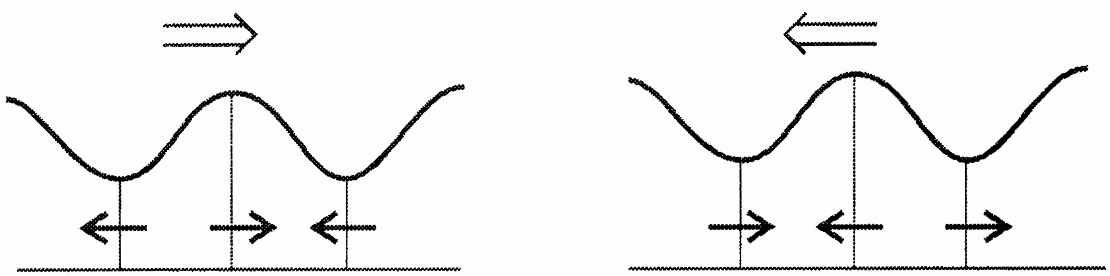

Figure 1.11: The relation between height and velocity perturbations for surface waves propagating to the right and left.

This means that convergence is occurring to the right of the height maximum for the wave propagating to the right, and to the left of the height maximum for one propagating to the left, which of course is consistent with the sense of propagation. (Note that the advective term in (1.1) vanished when we linearized.)

1.5 Surface wave reflection and modes

1.5.1 1-D reflection

In general, $H_\pm$ in eq. (1.16) are arbitrary constants, to be determined by initial and boundary conditions. Of course, if the waves are propagating boundary itself, therefore, while both waves are present there, the two waves [see eq. (1.16)] interfere constructively in the h field while they interfere destructively in u. So the height field perturbation actually amplifies while the wave packet is close to the boundary (Fig. 1.12(b)). (If you think this is getting something for nothing, note that the wave packet becomes laterally compressed during this time.) Subsequently, the entire incoming wave has been reflected and the wave packet propagates away to the left (c).

1.5.2 Modes in a bounded 1-D domain

Consider now a bounded domain, with coasts at $x = 0$ and $x = L$, at each of which $u' = 0$. The solutions (1.16) and (1.17) that satisfy these boundary conditions are

$$\begin{aligned} h'(x,t) &= H \cos kx \cos(kc_0 t - \epsilon) \; ; \\ u'(x,t) &= \frac{c_0}{D} H \sin kx \sin(kc_0 t - \epsilon) \; ; \end{aligned} \tag{1.18}$$

where H is a real constant amplitude and ϵ is an arbitrary constant phase (which could be eliminated by a choice of origin for t). This is a solution *provided* $u'(L,t) = 0$, which requires the modal condition that the wavenumber satisfy $k = k_n$, where

$$k_n = n\frac{\pi}{L} \; , \tag{1.19}$$

where n is a nonzero integer; the wave has n half-wavelengths across the domain. Thus, the allowable wavenumber spectrum is quantized, as is the allowable frequency spectrum:

$$\omega_n = k_n c_0 = n\frac{\pi c_0}{L} \; .$$

$$\begin{aligned} u'(0,t) &= \tfrac{c_0}{D} Re\left[H_+ e^{-ikc_0 t} - H_+^* e^{+ikc_0 t)}\right] \\ &= \tfrac{c_0}{D} Re\left[\left(H_+ e^{-ikc_0 t}\right) - \left(H_+ e^{-ikc_0 t}\right)^*\right] = 0 \; , \end{aligned}$$

since $Re\,(a - a^*) = 0$ for any a.

in an unbounded domain, the location of the sources will tell us which is nonzero (*e.g.*, if the only source for the wave is to the left, $H_- = 0$).

If, however, the domain is bounded, the wave may be reflected from the boundaries. Consider the semi-infinite domain bounded at its eastern side by a vertical coast at $x = 0$ (Fig. 1.12).

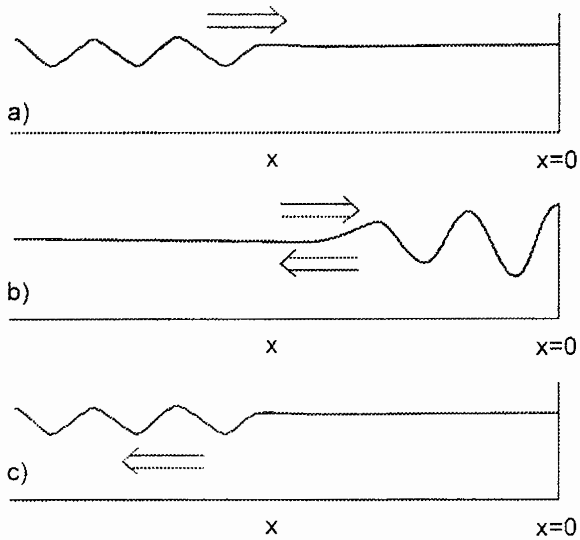

Figure 1.12: A shallow water wave reflecting from an eastern boundary.

A wave packet[4] generated at large negative x propagates in toward the boundary (a). According to Fig. 1.11, it has a nonzero u component in the peaks and troughs. After the wave has reached the boundary (b), it has to meet the boundary condition of zero motion normal to the coast (*i.e.*, $u = 0$ for all t), which a single wave component cannot do. The only way for the boundary condition to be met is for a second wave to be radiated from the boundary; in order for the u component of this wave to cancel that of the incoming wave at the boundary *at all times*, it must have the same magnitude of frequency and therefore, from (1.10), the same wavenumber. In short, it must be the mirror-image wave, propagating to the left, with equal and opposite amplitude to that of the incoming wave. In terms of (1.17), $H_- = H_+^*$, where the asterisk denotes complex conjugate[5]. At the

[4]By "wave packet", I mean a wave with a finite number of, but many, wavelengths.

[5]This guarantees $u = 0$ at the boundary $x = 0$, since then

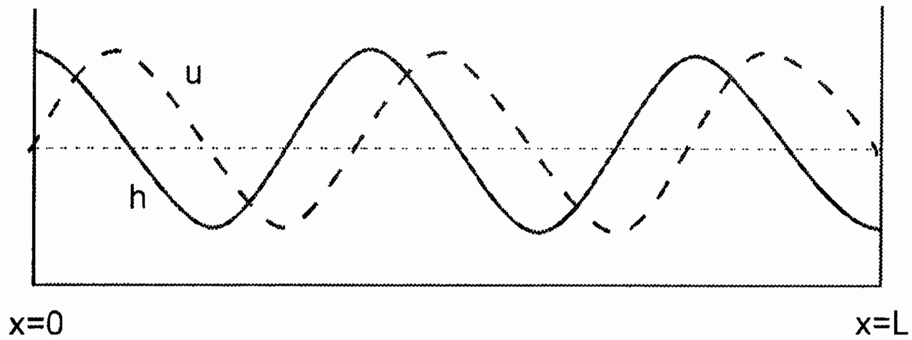

Figure 1.13: The n=5 1-D mode in a bounded domain.

So a finite 1-D domain of width L supports a countably infinite number of discrete modes, the lowest frequency of which is $\pi c_0/L$, or a maximum period of $2L/c_0$. Fig. 1.13 shows the u and h structure of the $n = 5$ mode; the patterns oscillate without propagation.

Of course, these standing wave modes can, in terms of (1.16) and (1.17), simply be regarded as sums of two equal and opposite propagating waves, continuously being reflected from the boundaries.

1.5.3 Reflection of plane waves

Reflection of plane waves is only slightly less straightforward than that of 1-D waves. At a straight boundary they suffer specular reflection (equal angles of incidence and reflection), as shown in Fig. 1.14.

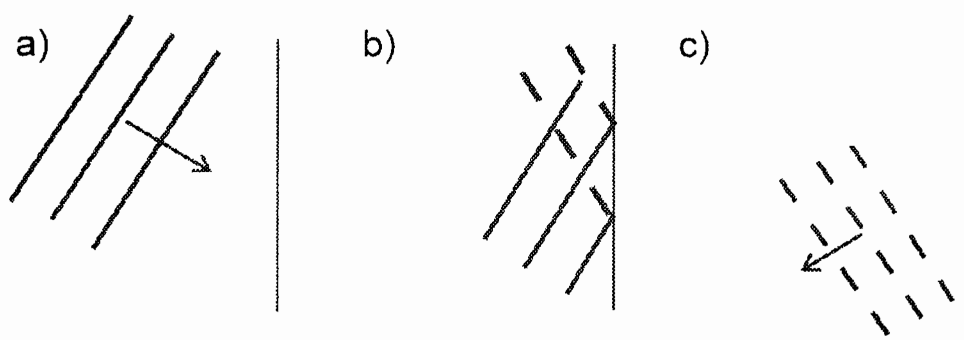

Figure 1.14: Reflection of plane waves.

[Note the interference between incident and reflected waves in (b). It is easy to generate almost-plane waves in a container (or a bathtub) and to see this effect.]

Modes also exist in 2-D containers with simple geometry (*e.g.*, rectangular or circular). In a rectangular basin of dimensions (L_x, L_y), modes are found with wavenumber components

$$k_m = m\frac{\pi}{L_x} \; ; \; l_n = n\frac{\pi}{L_y}$$

(where either m or n, but not both, can be zero) with corresponding allowable frequencies

$$\omega_{mn} = c_0\sqrt{k_m^2 + l_n^2} \; .$$

Fig, 1.15 shows a (3,2) mode, which has period

$$\frac{2\pi}{\omega_{32}} = \frac{2L_xL_y}{c_0\sqrt{9L_y^2 + 4L_x^2}} \; .$$

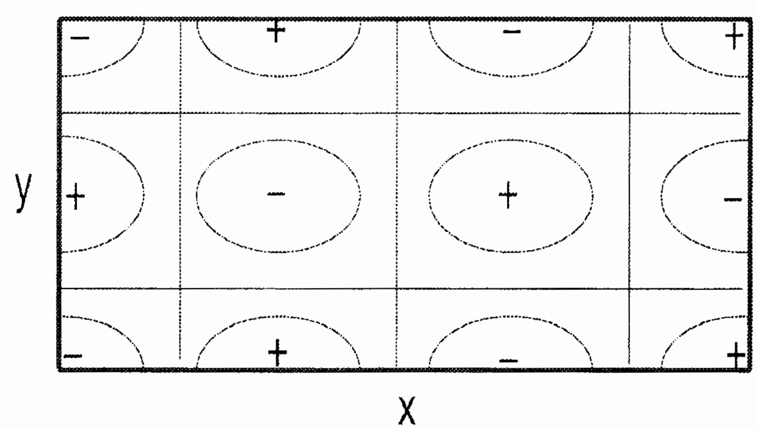

Figure 1.15: The structure of surface height displacement in a (3,2) mode in a rectangular basin.

1.6 Further reading

Elementary but brief descriptions of water waves (not confined to those on shallow water) can be found in:
Waves, tides and shallow-water processes, by the Open University Course Team, The Open University, Pergamon Press, 1989 (Chapter 1).
Elementary Fluid Dynamics, by D.J. Acheson, Clarendon Press, Oxford, 1990.

Other treatments can be found in many fluid dynamics texts, but are usually *much* more advanced and more mathematical than these two. One particularly thorough treatment is in
Waves in Fluids, by James Lighthill, Cambridge University Press, 1978 (Chapter 3).