Lecture notes for 12.009, Theoretical Environmental Analysis

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Contents

1 Spectral analysis

References: Bracewell [1], Muller and MacDonald [2], Berge et al. [3].

Astronomically forced phenomena such as glacial cycles give rise to signals in which periodic phenomena are superimposed on other types of variations.

One often seeks to measure the frequency of the various periodic components along with their relative amplitude.

To do so, we compute *power spectra*, using *Fourier transforms*.

These lectures are intended to provide a theoretical understanding of power spectra.

But we first consider some typical data.

1.1 Climatic signals

Reference: Emerson and Hedges [4], Muller and Macdonald [2].

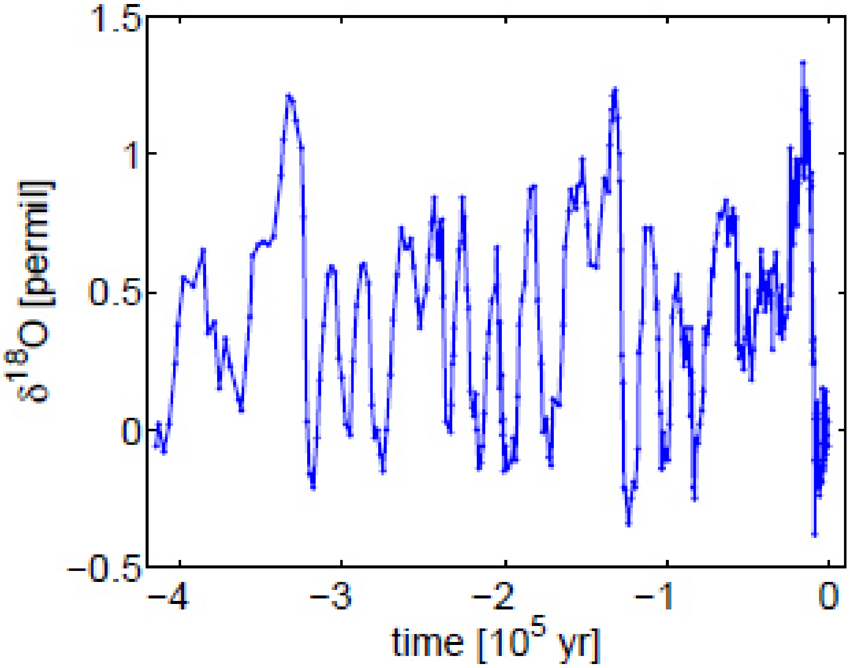
Ocean sediments and ice cores contain within them a great many signals of climate change, e.g.

- Isotopic composition of oxygen, which is sensitive to global ice volume and temperature, obtained from
  - entrapped air in ice cores; and
  - carbonate shells of planktic (sea surface) and benthic (sea bottom) organisms.
- Deuterium/hydrogen ratios (D/H), D = ^{2}H (^{1}H with a neutron), sensitive to temperature, in ice cores.
- Carbon isotopic compositions.
- Dust content, etc.

Perhaps the most studied signal is

$$\delta^{18}\mathrm{O} = \left(\frac{\left(^{18}\mathrm{O}/^{16}\mathrm{O}\right)_{\mathrm{sample}}}{\left(^{18}\mathrm{O}/^{16}\mathrm{O}\right)_{\mathrm{std}}} - 1 \right) \times 1000$$

Here's an example, from entrapped air in the Vostok (Antarctica) ice core (data from Petit et al. [5]):



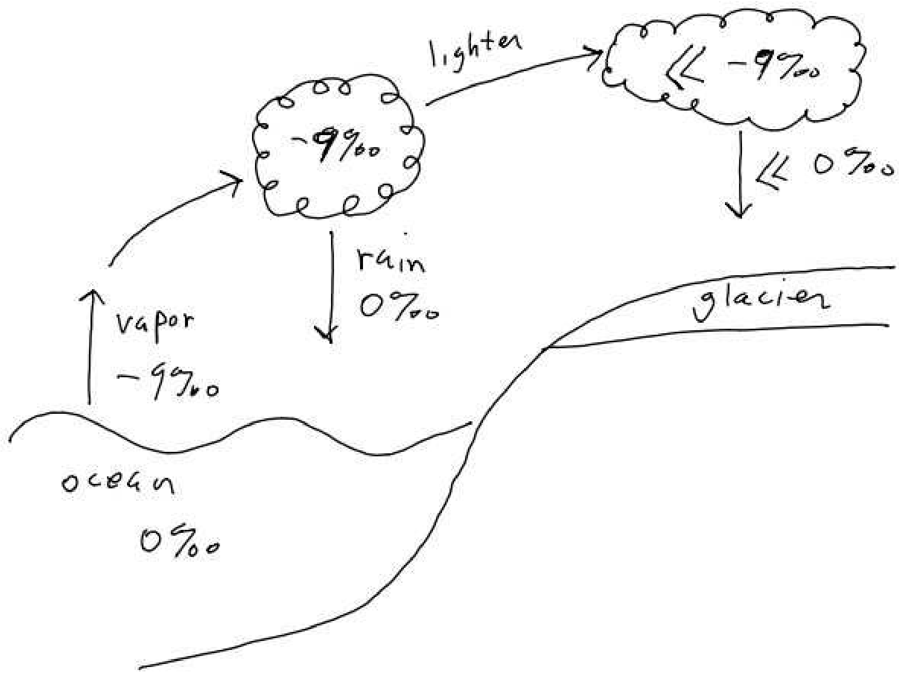
Note the clear occurrence of the precession signal, with a period of about 23 Kyr.

Many processes cause δ^{18}O to change. The two most important are the following:

- The vapor pressure (related to evaporation rate) of water containing ^{16}O is higher than that of water containing ^{18}O. Thus ^{16}O evaporates more readily, and *evaporated water is depleted in* ^{18}O.
- Conversely, *precipitated water is enriched in* ^{18}O. In other words, $H_2{}^{18}O$ condenses at a faster rate than $H_2{}^{16}O$.

The combined effect of these two processes leads to an enrichment of δ^{18}O of air as the global volume of ice grows.

Here's why:



Take ocean water to be at 0‰.

Evaporated water is typically about 9‰ lighter (at 20 °C) than liquid water.

Conversely condensate is about 9‰ heavier than vapor.

So a cloud forming from recently evaporated seawater has $\delta_{\text{cloud}} = -9‰$.

And the first rain from this cloud is 9‰ is heavier, so that $\delta_{\text{rain}} = 0‰$.

However the remaining water in the cloud must be isotopically lighter than it was originally, and the rain out of it will therefore also become lighter.

As the cloud moves to higher elevations or higher latitudes, it loses more vapor to condensate, and the resulting rain or snow becomes lighter and lighter.

This process, in which a particular mass—here a cloud—is progressively "milked" of the heavy isotope so that it becomes lighter and lighter (or vice-versa), is an example of *Rayleigh distillation.*

In the arctic, both clouds and the resulting snow are very light, less than

$-30‰$. The isotopic composition of the entire pool of condensate, from beginning to end, is of course equal to the original $-9‰$—but since the initial (low-latitude) precipitation is heavier, the final (high-latitude) precipitation must be much lighter.

Polar ice turns out to be about $-40‰$, i.e., about 4% lighter than the O_2 of seawater.

We also know that sea level was about 100 m lower during times of peak glaciation.

Since the average depth of the oceans is about 3800 m,

$$\frac{\text{total ice volume}}{\text{ice + ocean volume}} = \frac{100}{3800} = 2.6\%$$

Conservation of mass then requires that the isotopic composition δ_w of the remaining seawater satisfy

$$\frac{1}{38}(-40‰) + \frac{37}{38}\delta_w = 0$$

implying that

$$\delta_w \simeq 1.1‰.$$

The O_2 in the atmosphere is created by photosynthesis, from water:

$$CO_2 + H_2O \to CH_20 + O_2.$$

The $\delta^{18}O$ of entrapped air in ice cores should therefore change more or less as the $\delta^{18}O$ of seawater, i.e., it should be about $1‰$ heavier in glacial times than interglacials, just as seen in the Vostok ice core.

1.2 Fourier transforms

The precise oscillatory nature of an observed time series $x(t)$ is usually not identifiable from $x(t)$ alone.

We may ask

- How well-defined is the the dominant frequency of oscillation?
- How many frequencies of oscillation are present?
- What are the relative contributions of all frequencies?

The analytic tool for answering these and myriad related questions is the *Fourier transform.*

1.2.1 Continuous Fourier transform

We first state the Fourier transform for functions that are continuous with time.

The Fourier transform of a function $f(t)$ is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \mathrm{d}t$$

Similarly, the inverse Fourier transform is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \mathrm{d}\omega.$$

That the second relation is the inverse of the first may be proven, but we save that calculation for the discrete transform, below.

1.2.2 Discrete-time signals

We are interested in the analysis of observational or experimental data, which is almost always discrete. Thus we specialize to *discrete Fourier transforms.*

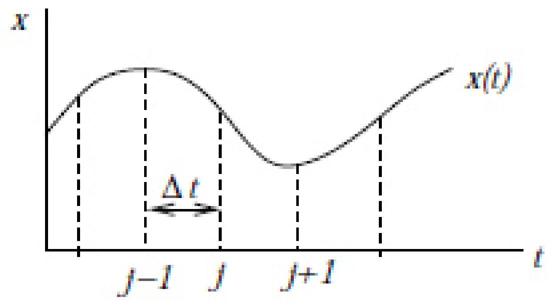
In modern data, one almost always observes a discretized signal

$$x_j, \qquad j = \{0, 1, 2, \dots, n-1\}$$

We take the *sampling interval*—the time between samples—to be Δt. Then

$$x_j = x(j\Delta t).$$

The discretization process is pictured as

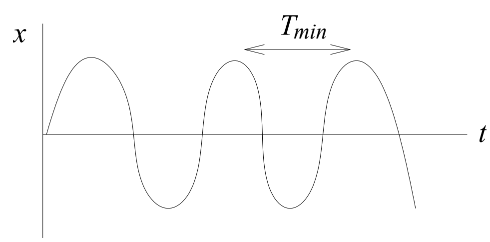


A practical question concerns the choice of Δt. To choose it, we must know the highest frequency, $f_{\max}$, contained in $x(t)$.

The shortest period of oscillation is

$$T_{\min} = 1/f_{\max}$$

Pictorially,



We require at least two samples per period. Therefore

$$\Delta t \leq \frac{T_{\min}}{2} = \frac{1}{2f_{\max}}.$$

To see why, we note that if a continuous signal $f(t)$ contains no frequencies greater than $f_{\max}$, the inverse Fourier transform of $F(\omega)$ may be written

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi f_{\max}}^{2\pi f_{\max}} F(\omega) e^{i\omega t} \mathrm{d}\omega$$

since $F(\omega) = 0$ when $|\omega| > 2\pi f_{\max}$.

Now define the jth sampling time

$$t_j = j\frac{T_{\min}}{2} = \frac{j}{2f_{\max}}, \qquad j = \ldots -1, 0, 1\ldots$$

Substituting t_j for t above, we obtain

$$f(t_j) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi f_{\max}}^{2\pi f_{\max}} F(\omega) e^{i\omega t_j} \mathrm{d}\omega.$$

The RHS is the jth coefficient in a Fourier-series expansion of $F(\omega)$.

Consequently the sampled function $f(t_j)$ completely determines $F(\omega)$.

And by inverse Fourier transformation, the continuous function $f(t)$ is completely determined by $F(\omega)$.

This reasoning, first given by Shannon [6], leaves open the question of how to reconstruct the continuous function $f(t)$ when only $f(t_j)$ is known.

In principle, exact interpolation is possible via the convolutional sum

$$f(t) = \sum_{j=-\infty}^{\infty} x_j \frac{\sin \pi(2f_{\max}t - j)}{\pi(2f_{\max}t - j)}.$$

where $x_j = f(t_j)$.

1.2.3 Discrete Fourier transform

The discrete Fourier transform (DFT) of a time series $x_j, j = 0, 1, \ldots, n-1$ is

$$\hat{x}_k = \sum_{j=0}^{n-1} x_j \exp\left(-i\frac{2\pi jk}{n}\right) \qquad k = 0, 1, \ldots, n-1$$

To gain some intuitive understanding, consider the range of the exponential multiplier.

- $k = 0 \Rightarrow \exp(-i2\pi jk/n) = 1$. Then
$$\hat{x}_0 = \sum_j x_j$$

Thus $\hat{x}_0$ is n times the mean of the x_j's.

This is the "DC" component of the transform.

Question: Suppose a seismometer measures ground motion. What would $\hat{x}_0 \neq 0$ mean?

- $k = n/2 \Rightarrow \exp(-i2\pi jk/n) = \exp(-i\pi j)$. Then

$$\begin{aligned}\hat{x}_{n/2} &= \sum_j x_j(-1)^j \\ &= x_0 - x_1 + x_2 - x_3 \ldots\end{aligned}$$

  Frequency index $n/2$ is clearly the highest accessible frequency.

- The frequency indices $k = 0, 1, \ldots, n/2$ correspond to frequencies

$$f_k = k/t_{\max},$$

  i.e., k oscillations per $t_{\max}$, the period of observation.
  Index $k = n/2$ then corresponds to

$$f_{\max} = \left(\frac{n}{2}\right)\left(\frac{1}{n\Delta t}\right) = \frac{1}{2\Delta t}$$

But if $n/2$ is the highest frequency that the signal can carry, what is the significance of $\hat{x}_k$ for $k > n/2$?

For real x_j, frequency indicies $k > n/2$ are *redundant*, being related by

$$\hat{x}_k = \hat{x}^*_{n-k}$$

where z^* is the complex conjugate of z (i.e., if $z = a + ib$, $z^* = a - ib$).

We derive this relation as follows. From the definition of the DFT, we have

$$\begin{aligned}
\hat{x}^*_{n-k} &= \sum_{j=0}^{n-1} x_j \exp\left(+i\frac{2\pi j(n-k)}{n}\right) \\
&= \sum_{j=0}^{n-1} x_j \underbrace{\exp(i2\pi j)}_{1} \exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \sum_{j=0}^{n-1} x_j \exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \hat{x}_k
\end{aligned}$$

where the + in the first equation derives from the complex conjugation, and the last line again employs the definition of the DFT.

Note that we also have the relation

$$\hat{x}^*_{-k} = \hat{x}^*_{n-k} = \hat{x}_k.$$

The frequency indicies $k > n/2$ are therefore sometimes referred to as *negative frequencies*

1.2.4 Inverse discrete Fourier tranform

The inverse DFT is given by

$$x_j = \frac{1}{n}\sum_{k=0}^{n-1} \hat{x}_k \exp\left(+i\frac{2\pi jk}{n}\right) \qquad j = 0, 1, \ldots, n-1$$

We proceed to demonstrate this inverse relation.

We begin by substituting the DFT for $\hat{x}_k$, using dummy variable j':

$$\begin{aligned}
x_j &= \frac{1}{n}\sum_{k=0}^{n-1}\left[\sum_{j'=0}^{n-1} x_{j'} \exp\left(-i\frac{2\pi j' k}{n}\right)\right]\exp\left(+i\frac{2\pi k j}{n}\right) \\
&= \frac{1}{n}\sum_{j'=0}^{n-1} x_{j'} \sum_{k=0}^{n-1}\exp\left(-i\frac{2\pi k(j'-j)}{n}\right) \\
&= \frac{1}{n}\sum_{j'=0}^{n-1} x_{j'} \times \begin{cases} n, & j'=j \\ 0, & j'\neq j \end{cases} \\
&= \frac{1}{n}(n x_j) \\
&= x_j
\end{aligned}$$

The third relation derives from the fact that the previous $\sum_k$ amounts to a vanishing sum over the unit circle in the complex plane, except when $j' = j$.

To see why the sum over the circle vanishes, consider the example of

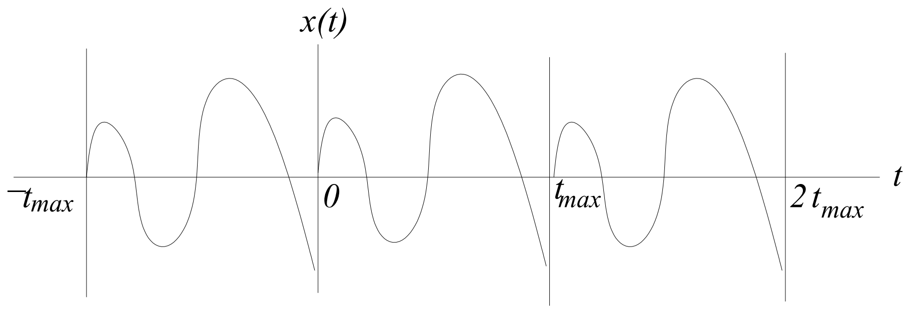
$$j' - j = 1 \quad \text{and} \quad n = 4.$$

The elements of the sum are then just the four points on the unit circle that intersect the real and imaginary axes, i.e.,

$$\begin{aligned}
\sum_{k=0}^{3}\exp\left(-i\frac{2\pi k(j'-j)}{4}\right) &= e^0 + e^{-i\pi/2} + e^{-i\pi} + e^{-i3\pi/2} \\
&= 1 + i - 1 - i \\
&= 0.
\end{aligned}$$

Finally, note that the DFT relations imply that x_j is periodic in n, so that $x_{j+n} = x_j$.

Consequently a finite time series is treated as if it were recurring:



1.3 The autocorrelation function and the power spectrum

Assume that the time series x_j has zero mean and that it is periodic, i.e., $x_{j+n} = x_j$.

Define the *autocorrelation function* ψ:

$$\psi_m = \sum_{j=0}^{n-1} x_j^* x_{j+m}$$

where

$$\psi_m = \psi(m\Delta t)$$

The autocorrelation function measures the degree to which a signal resembles itself over time. Thus it measures the predictability of the future from the past.

To gain some intuition:

- Consider, for example, $m = 0$ and real x_j. Then

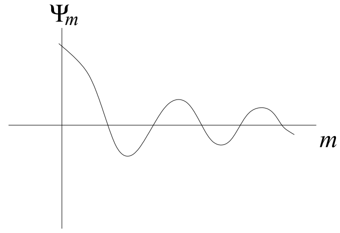
  $$\psi_0 = \sum_{j=0}^{n-1} x_j^2,$$

  which is n times the mean squared value of x_j.

- Alternatively, if $m\Delta t$ is much less than the dominant period of the data, ψ_m should not be too much less than ψ_0.

- Last, if $m\Delta_t$ is much greater than the dominant period of the data, $|\psi_m|$ is relatively small.

A typical ψ_m looks like



The *power spectrum* of a time series is the magnitude squared of its Fourier transform:

$$|\hat{x}_k|^2 = \left| \sum_{j=0}^{n-1} x_j \exp\left(-i\frac{2\pi jk}{n}\right) \right|^2 .$$

The *Wiener-Khintchin theorem* states that

power spectrum = Fourier transform of the autocorrelation.

In symbols,

$$|\hat{x}_k|^2 = \sum_{m=0}^{n-1} \psi_m \exp\left(-i\frac{2\pi km}{n}\right)$$

We also have the inverse relation

$$\psi_m = \frac{1}{n} \sum_{k=0}^{n-1} |\hat{x}_k|^2 \exp\left(+i\frac{2\pi km}{n}\right)$$

To prove the latter relation, we first substitute the inverse DFT for x_j and x_{j+m} in the definition of ψ_m:

$$\begin{aligned}
\psi_m &= \sum_{j=0}^{n-1} x_j^* x_{j+m} \\
&= \sum_{j=0}^{n-1} \left[\frac{1}{n} \sum_{k=0}^{n-1} \hat{x}_k^* \exp\left(-i\frac{2\pi kj}{n}\right) \right] \left[\frac{1}{n} \sum_{k'=0}^{n-1} \hat{x}_{k'} \exp\left(i\frac{2\pi k'(j+m)}{n}\right) \right]
\end{aligned}$$

We then change the order of the summations and simplify as follows:

$$\psi_m = \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1}\hat{x}_k^*\hat{x}_{k'}\exp\left(i\frac{2\pi mk'}{n}\right)\underbrace{\sum_{j=0}^{n-1}\exp\left(i\frac{2\pi j(k'-k)}{n}\right)}_{\substack{=n,\quad k'=k\\ =0,\quad k'\neq k}}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\hat{x}_k^*\hat{x}_k\exp\left(i\frac{2\pi mk}{n}\right)$$

which is the Wiener-Khintchin relation.

By Fourier transforming ψ_m we also prove the inverse relation: the power spectrum is the Fourier transform of the autocorrelation.

For a real time series $\{x_j\}$, we can use the previously derived relation

$$\hat{x}_k^* = \hat{x}_{n-k} = \hat{x}_{-k}$$

to show that

$$|\hat{x}_k|^2 = \hat{x}_k\hat{x}_k^* = \hat{x}_k\hat{x}_{n-k} = \hat{x}_{n-k}^*\hat{x}_{n-k} = |\hat{x}_{n-k}|^2.$$

This redundancy results from the fact that neither the autocorrelation nor the power spectrum contain information on any "phase lags" in either x_j or its individual frequency components.

Thus while the DFT of an n-point time series results in n independent quantities ($2\ \times n/2$ complex numbers), the power spectrum yields only $n/2$ independent quantities.

One may therefore show that there are an infinite number of time series that have the same power spectrum, but that each time series uniquely defines its Fourier transform, and vice-versa.

Consequently a time series cannot be reconstructed from its power spectrum or autocorrelation function.

1.4 Power spectrum of a periodic signal

Consider a periodic signal

$$x(t) = x(t+T) = x\left(t + \frac{2\pi}{\omega}\right)$$

Consider the extreme case where the period T is equal to the duration of the signal:

$$T = t_{\max} = n\triangle t$$

The Fourier components are separated by

$$\Delta f = \frac{1}{t_{\max}}$$

i.e. at frequencies

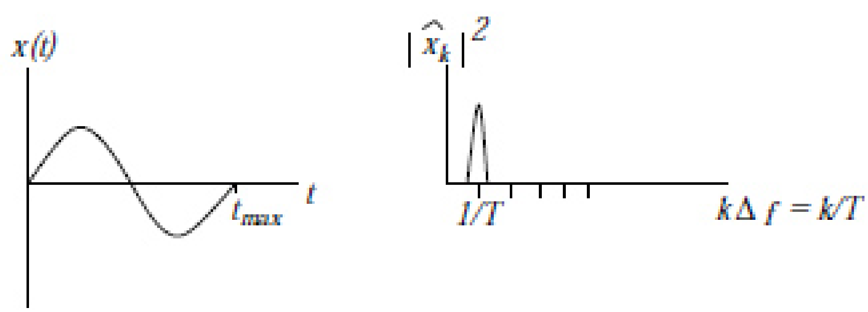
$$0,\ 1/T,\ 2/T, \ldots, (n-1)/T.$$

1.4.1 Sinusoidal signal

In the simplest case, $x(t)$ is a sine or cosine, i.e.,

$$x(t) = \sin\left(\frac{2\pi t}{t_{\max}}\right).$$

What is the Fourier tranform? Pictorially, we expect



We calculate the power spectrum analytically, beginning with the DFT:

$$
\begin{aligned}
\hat{x}_k &= \sum_j x_j \exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \sum_j \sin\left(\frac{2\pi j\Delta t}{t_{\max}}\right)\exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \frac{1}{2i}\sum_j \left[\exp\left(\frac{i2\pi j\Delta t}{t_{\max}}\right) - \exp\left(\frac{-i2\pi j\Delta t}{t_{\max}}\right)\right]\exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \frac{1}{2i}\sum_j \left[\exp\left\{i2\pi j\left(\frac{\Delta t}{t_{\max}} - \frac{k}{n}\right)\right\} - \exp\left\{-i2\pi j\left(\frac{\Delta t}{t_{\max}} + \frac{k}{n}\right)\right\}\right] \\
&= \pm\frac{n}{2i} \qquad \text{when } k = \frac{\pm n\Delta t}{t_{\max}}.
\end{aligned}
$$

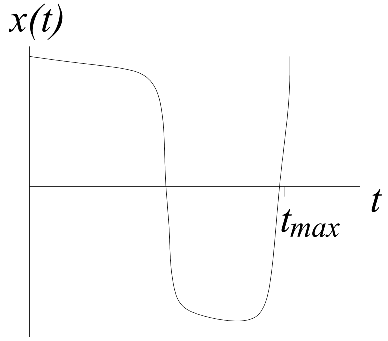
Thus

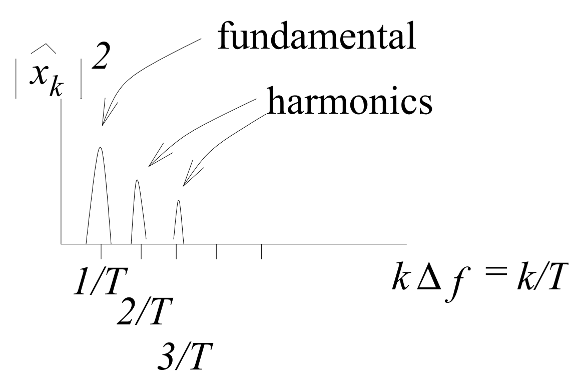
$$
|\hat{x}_k|^2 = \frac{n^2}{4} \qquad \text{for } k = \pm 1.
$$

1.4.2 Non-sinusoidal signal

Consider now a non-sinusoidal yet periodic signal, similar to that of the signals seen in glacial cycles.

The non-sinusoidal character of such oscillations implies that it contains higher-order *harmonics*, i.e., integer multiples of the *fundamental frequency* $1/T$. Thus, pictorially, we expect





Now suppose $t_{\max} = pT$, where p is an integer. The non-zero components of the power spectrum must still be at frequencies

$$1/T,\ 2/T, \ldots .$$

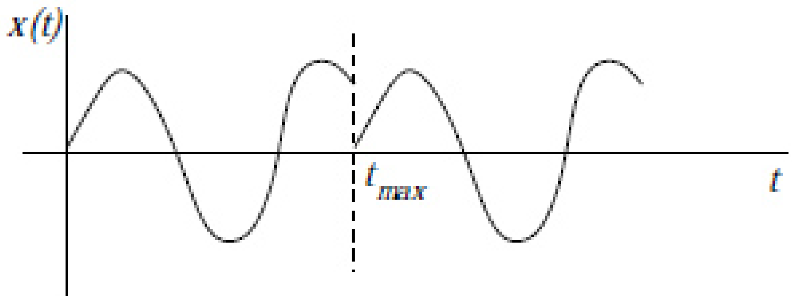
But since

$$\Delta f = \frac{1}{t_{\max}} = \frac{1}{pT}$$

the frequency resolution is p times greater. Contributions to the power spectrum would remain at integer multiples of the frequency $1/T$, but spaced p samples apart on the frequency axis.

1.4.3 $t_{\max}/T \neq$ integer

If $t_{\max}/T$ is not an integer, the (effectively periodic) signal looks like



We calculate the power spectrum of such a signal, assuming the sinusoidal function

$$x(t) = \exp\left(i\frac{2\pi t}{T}\right)$$

which has the discrete form

$$x_j = \exp\left(i\frac{2\pi j\Delta t}{T}\right).$$

The DFT is

$$\hat{x}_k = \sum_{j=0}^{n-1} \exp\left(i\frac{2\pi j\Delta t}{T}\right) \exp\left(-i\frac{2\pi jk}{n}\right).$$

Set

$$\phi_k = \frac{\Delta t}{T} - \frac{k}{n}.$$

Then

$$\hat{x}_k = \sum_{j=0}^{n-1} \exp\left(i2\pi\phi_k j\right).$$

Recall the identity

$$\sum_{j=0}^{n-1} x^j = \frac{x^n - 1}{x - 1}.$$

Then

$$\hat{x}_k = \frac{\exp(i2\pi\phi_k n) - 1}{\exp(i2\pi\phi_k) - 1}.$$

The power spectrum is

$$\begin{aligned}
|\hat{x}_k|^2 = \hat{x}_k\hat{x}_k^* &= \frac{1 - \cos(2\pi\phi_k n)}{1 - \cos(2\pi\phi_k)} \\
&= \frac{\sin^2(\pi\phi_k n)}{\sin^2(\pi\phi_k)}.
\end{aligned}$$

Note that

$$n\phi_k = \frac{n\Delta t}{T} - k = \frac{t_{\max}}{T} - k$$

is the difference between a DFT index k and the "real" non-integral frequency index $t_{\max}/T$.

Assume that n is large and k is close to that "real" frequency index such that

$$n\phi_k = \frac{n\Delta t}{T} - k \ll n.$$

Consequently $\phi_k \ll 1$, so we may also assume

$$\pi\phi_k \ll 1.$$

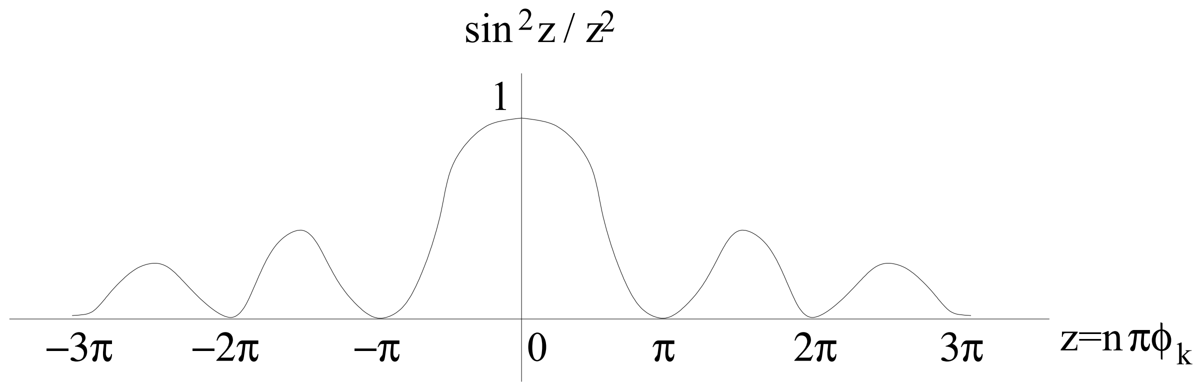
Then

$$\begin{aligned}
|\hat{x}_k|^2 &\simeq \frac{\sin^2(\pi\phi_k n)}{(\pi\phi_k)^2} \\
&= n^2\,\frac{\sin^2(\pi\phi_k n)}{(\pi\phi_k n)^2} \\
&\propto \frac{\sin^2 z}{z^2}
\end{aligned}$$

where

$$z = n\pi\phi_k = \pi\left(\frac{n\Delta t}{T} - k\right) = \pi\left(\frac{t_{\max}}{T} - k\right).$$

Thus $|\hat{x}_k|^2$ is no longer a simple spike. Instead, as a function of $z = n\pi\phi_k$ it appears as



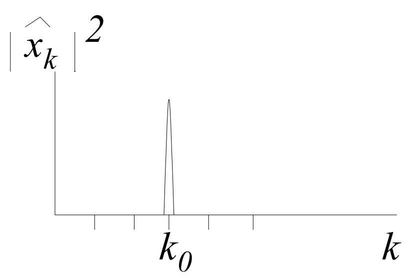
The plot gives the kth component of the power spectrum of $e^{i2\pi t/T}$ as a function of $\pi(t_{\max}/T - k)$.

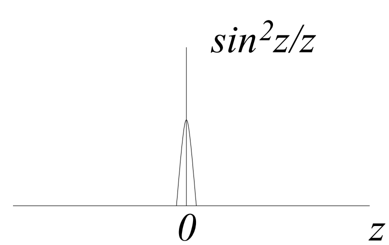
To interpret the plot, let k_0 be the integer closest to $t_{\max}/T$. There are then two extreme cases:

1. $t_{\max}$ is an integral multiple of T:

   $$\frac{t_{\max}}{T} - k_0 = 0.$$

   The spectrum is perfectly sharp:

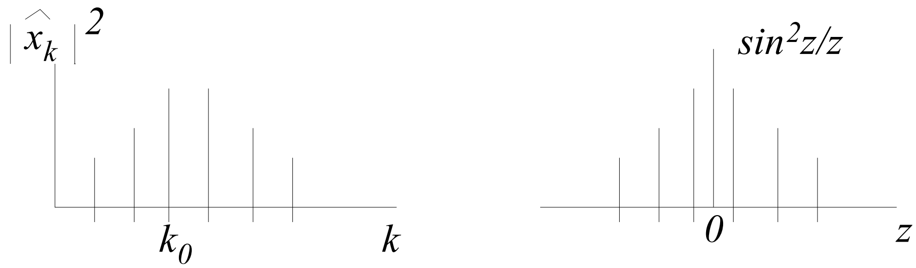
   

2. $t_{\max}/T$ falls midway between two frequencies. Then

   $$\frac{t_{\max}}{T} - k_0 = \frac{1}{2}.$$

   The spectrum is smeared:



The smear decays like

$$\frac{1}{(k - t_{\max}/T)^2} \sim \frac{1}{k^2}$$

1.4.4 Conclusion

The power spectrum of a periodic signal of period T is composed of:

1. a peak at the frequency $1/T$
2. a smear (sidelobes) near $1/T$
3. possibly harmonics (integer multiples) of $1/T$
4. smears near the harmonics.

1.5 Quasiperiodic signals

Let y be a function of r independent variables:

$$y = y(t_1, t_2, \ldots, t_r).$$

y is *periodic*, of period 2π in *each* argument, if

$$y(t_1, t_2, \ldots, t_j + 2\pi, \ldots, t_r) = y(t_1, t_2, \ldots, t_j, \ldots, t_r), \quad j = 1, \ldots, r$$

y is called *quasiperiodic* if each t_j varies with time at a different rate (i.e., different "clocks"). We have then

$$t_j = \omega_j t, \qquad j = 1, \ldots, r.$$

The quasiperiodic function y has r fundamental frequencies:

$$f_j = \frac{\omega_j}{2\pi}$$

and r periods

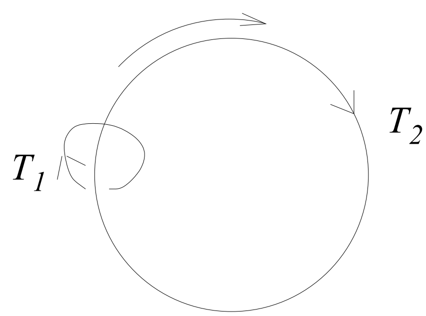
$$T_j = \frac{1}{f_j} = \frac{2\pi}{\omega_j}.$$

Example: The astronomical position of a point on Earth's surface changes due to

- rotation of Earth about axis ($T_1 = 24$ hours).
- revolution of Earth around sun ($T_2 \simeq 365$ days).

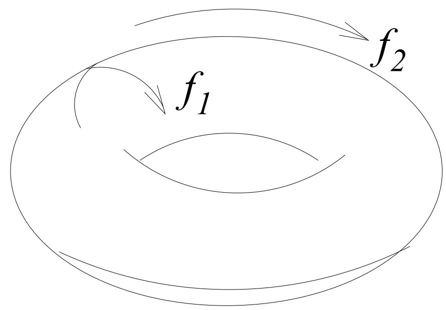
At long time scales, we also have changes in precession (26 Kyr), obliquity (41 Kyr), and eccentricity (∼100 Kyr).

Considering just two oscillations (e.g, rotation and revolution), we can conceive of such a function on a 2-D torus T^2, existing in a 3-D space.



Here we think of a disk spinning with period T_1 while it revolves along the circular path with period T_2.

Such behavior can be conceived as a trajectory on the *surface* of a doughnut or inner tube, or a torus T_2 in $\mathbb{R}^3$.



What is the power spectrum of a quasiperiodic signal $x(t)$? There are two possibilities:

1. The quasiperiodic signal is a *linear* combination of independent periodic functions. For example:

$$x(t) = \sum_{i=1}^{r} x_i(\omega_i t).$$

   Because the Fourier transform is a linear transformation, the power spectrum of $x(t)$ is a set of peaks at frequencies

$$f_1 = \omega_1/2\pi,\ f_2 = \omega_2/2\pi, \ldots$$

   and their harmonics

$$m_1 f_1,\ m_2 f_2, \ldots \qquad (m_1, m_2, \ldots \text{ positive integers}).$$

2. The quasiperiodic signal $x(t)$ depends nonlinearly on periodic functions. For example,

$$x(t) = \sin(2\pi f_1 t)\sin(2\pi f_2 t) = \frac{1}{2}\cos(|f_1 - f_2|2\pi t) - \frac{1}{2}\cos(|f_1 + f_2|2\pi t).$$

   The fundamental frequencies are

$$|f_1 - f_2| \qquad \text{and} \qquad |f_1 + f_2|.$$

   The harmonics are

$$m_1|f_1 - f_2| \qquad \text{and} \qquad m_2|f_1 + f_2|, \qquad m_1, m_2 \text{ positive integers}.$$

The nonlinear case requires more attention. In general, if $x(t)$ depends nonlinearly on r periodic functions, then the harmonics are

$$|m_1 f_1 + m_2 f_2 + \ldots + m_r f_r|, \qquad m_i \text{ arbitrary integers.}$$

In what follows, we specialize to $r = 2$ frequencies, and forget about finite Δf.

Each nonzero component of the spectrum of $x(\omega_1 t, \omega_2 t)$ is a peak at

$$f = |m_1 f_1 + m_2 f_2|, \qquad m_1, m_2 \text{ integers .}$$

There are two cases:

1. f_1/f_2 rational $\Rightarrow$ *sparse spectrum.*
2. f_1/f_2 irrational $\Rightarrow$ *dense spectrum.*

To understand this, rewrite f as

$$f = f_2 \left| m_1 \frac{f_1}{f_2} + m_2 \right| .$$

In the rational case,

$$\frac{f_1}{f_2} = \frac{\text{integer}}{\text{integer}}.$$

Then

$$\left| m_1 \frac{f_1}{f_2} + m_2 \right| = \left| \frac{\text{integer}}{f_2} + \text{integer} \right| = \text{ integer multiple of } \frac{1}{f_2}.$$

Thus the peaks of the spectrum must be separated (i.e., sparse).

Alternatively, if f_1/f_2 is irrational, then m_1 and m_2 may always be chosen so that

$$\left| m_1 \frac{f_1}{f_2} + m_2 \right| \text{ is not similarly restricted.}$$

These distinctions have further implications.

In the rational case,

$$\frac{f_1}{f_2} = \frac{n_1}{n_2}, \qquad n_1, n_2 \text{ integers.}$$

Since

$$\frac{n_1}{f_1} = \frac{n_2}{f_2}$$

the quasiperiodic function is *periodic* with period

$$T = n_1 T_1 = n_2 T_2.$$

All spectral peaks must then be harmonics of the fundamental frequency

$$f_0 = \frac{1}{T} = \frac{f_1}{n_1} = \frac{f_2}{n_2}.$$

Thus the rational quasiperiodic case is in fact periodic, and some writers restrict quasiperiodicity to the irrational case.

Note further that, in the irrational case, the signal never exactly repeats itself.

One may consider, as an example, the case of a child walking on a sidewalk, attempting with uniform steps to never step on a crack (and breaking his mother's back...).

Then if $x(t)$ were the distance from the closest crack at each step, it would only be possible to avoid stepping on a crack if the ratio

$$\frac{\text{step size}}{\text{crack width}}$$

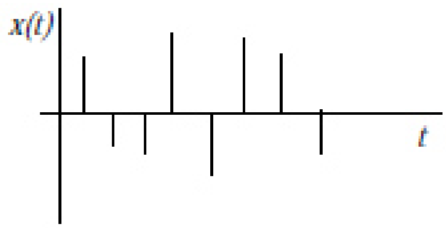
were rational.

1.6 Aperiodic signals

Aperiodic signals are neither periodic nor quasiperiodic.

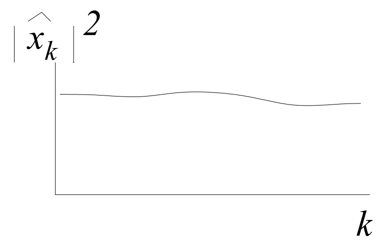
Aperiodic signals appear random, though they may have a deterministic foundation.

An example is white noise, which is a signal that is “new” and unpredictable at each instant, e.g.,



Statistically, each sample of a white-noise signal is independent of the others, and therefore uncorrelated to them.

The power spectrum of white noise is, on average, flat:



The flat spectrum of white noise is a consequence of its lack of harmonic structure (i.e., one cannot recognize any particular tone, or dominant frequency).

We proceed to derive the spectrum of a white noise signal $x(t)$.

Rather than considering only one white-noise signal, we consider an *ensemble* of such signals, i.e.,

$$x^{(1)}(t), x^{(2)}(t), \ldots$$

where the superscipt denotes the particular realization within the ensemble. Each realization is independent of the others.

Now discretize each signal so that

$$x_j = x(j\Delta t), \qquad j = 0, \ldots, n-1$$

We take the signal to have finite length n but consider the ensemble to contain an infinite number of realizations.

We use angle brackets to denote *ensemble averages*.

The ensemble-averaged mean of the jth sample is then

$$\langle x_j \rangle = \lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} x_j^{(i)}$$

Similarly, the mean-square value of the jth sample is

$$\left\langle x_j^2 \right\rangle = \lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} \left(x_j^{(i)} \right)^2$$

Now assume *stationarity*: $\langle x_j \rangle$ and $\left\langle x_j^2 \right\rangle$ are independent of j. We take these mean values to be $\langle x \rangle$ and $\left\langle x^2 \right\rangle$, respectively, and assume $\langle x \rangle = 0$.

Recall the autocorrelation ψ_m:

$$\psi_m = \sum_{j=0}^{n-1} x_j x_{j+m}.$$

By definition, each sample of white noise is uncorrelated with its past and future. Therefore

$$\begin{aligned} \langle \psi_m \rangle &= \left\langle \sum_j x_j x_{j+m} \right\rangle \\ &= n \left\langle x^2 \right\rangle \delta_m \end{aligned}$$

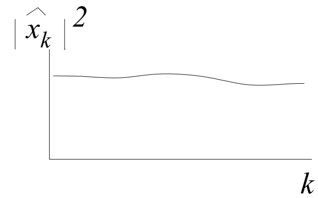
where

$$\delta_m = \begin{cases} 1 & m = 0 \\ 0 & \text{else} \end{cases}$$

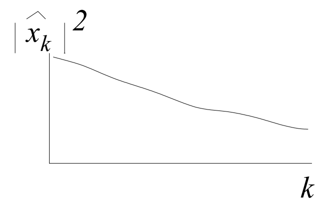
We obtain the power spectrum from the autocorrelation function by the Wiener-Khintchine theorem:

$$\begin{aligned} \left\langle |\hat{x}_k|^2 \right\rangle &= \sum_{m=0}^{n-1} \langle \psi_m \rangle \exp\left(-i \frac{2\pi mk}{n} \right) \\ &= \sum_{m=0}^{n-1} n \left\langle x^2 \right\rangle \delta_m \exp\left(-i \frac{2\pi mk}{n} \right) \\ &= n \left\langle x^2 \right\rangle \\ &= \text{constant.} \end{aligned}$$

Thus for white noise, the spectrum is indeed flat, as previously indicated:

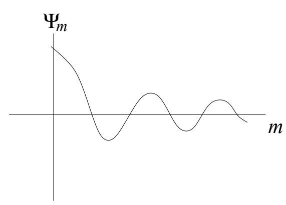


A more common case is "colored" noise: a continuous spectrum, but not constant:



In such (red) colored spectra, there is a relative lack of high frequencies. The signal is still apparently random, but only beyond some interval Δt.

The autocorrelation of colored noise is broader, e.g.,



1.7 Power spectrum of a random walk

Colored noise often has a power spectrum that decays like

$$\left\langle |\hat{x}_k|^2 \right\rangle \propto k^{-\beta}, \qquad \beta \simeq 2.$$

Here we show how the case $\beta = 2$ derives from a random walk.

Suppose that $\{x_j\}$ is a random walk. Then the increments

$$\eta_j = x_j - x_{j-1}$$

are all independent, so that the the autocorrelation

$$\begin{aligned}\langle \psi_m \rangle &= \sum_j \langle \eta_j \eta_{j+m} \rangle \\ &= n \left\langle \eta^2 \right\rangle \delta_m.\end{aligned}$$

Therefore the power spectrum of the increments is flat:

$$\left\langle |\hat{\eta}_k|^2 \right\rangle = \text{const.}$$

We can also calculate $\hat{\eta}_k$ from the Fourier transform of x_j:

$$\begin{aligned}\hat{\eta}_k &= \sum_j (x_j - x_{j-1}) \exp\left(-i\frac{2\pi jk}{n}\right) \\ &= \hat{x}_k - \sum_j x_j \exp\left(-i\frac{2\pi(j+1)k}{n}\right) \\ &= \hat{x}_k \left[1 - \exp\left(-i\frac{2\pi k}{n}\right)\right].\end{aligned}$$

where we have used the periodicity of x_j, i.e., $x_{j+n} = x_j$, in writing the summation in the second relation.

Squaring both sides above, we obtain the power spectrum

$$|\hat{\eta}_k|^2 = 2|\hat{x}_k|^2 \left[1 - \cos\left(\frac{2\pi k}{n}\right)\right].$$

Since

$$\cos x = 1 - \frac{x^2}{2} + \mathcal{O}(x^4)$$

we have, for $2\pi k \ll n$,

$$|\hat{\eta}_k|^2 \propto k^2 |\hat{x}_k|^2.$$

But we know that $\left\langle |\hat{\eta}_k|^2 \right\rangle = \text{const.}$ Therefore

$$\left\langle |\hat{x}_k|^2 \right\rangle \propto k^{-2}.$$

Thus when spectra decay like $1/k^2$, the underlying time series could be a random walk.

But beware: many other processes also give spectra that decay like $1/k^2$.

1.8 Identification of spectral peaks

Reference: Scargle [7].

Suppose you compute the DFT of a particular signal and identify a spectral peak. Is the peak real?

To answer this question, we need a null hypothesis and ask a specific question:

> *If a time series is composed of uncorrelated (white) noise, what is the probability of observing a spectral peak with a power greater than the power observed?*

Suppose that x_j is Gaussian white noise with zero mean and variance

$$\sigma^2 = \left\langle x_j^2 \right\rangle = \frac{1}{n} \sum_j x_j^2.$$

The DFT of x_j is

$$\begin{aligned} \hat{x}_k &= \sum_j x_j \exp\left(-i\frac{2\pi jk}{n}\right) \\ &\equiv a_k + ib_k. \end{aligned}$$

where

$$a_k = \sum_j x_j \cos\left(2\pi jk/n\right)$$

and

$$b_k = -\sum_j x_j \sin\left(2\pi jk/n\right)$$

Because the x_j are independent Gaussian random variables with zero mean, so too are a_k and b_k. The mean power spectrum S_k is

$$\begin{aligned} S_k = \left\langle |\hat{x}_k|^2 \right\rangle &= \left\langle a_k^2 + b_k^2 \right\rangle \\ &= \left\langle a_k^2 \right\rangle + \left\langle b_k^2 \right\rangle \\ &= n\sigma^2 \end{aligned}$$

where the latter relation was established in Section 1.6.

For evenly sampled data the variance is equally shared so that

$$\langle a_k^2 \rangle = \langle b_k^2 \rangle = \frac{n\sigma^2}{2} \equiv s_0^2$$

Consequently the probability density functions of a_k and b_k are Gaussian with zero mean and variance s_0^2:

$$p(a) = \frac{1}{\sqrt{2\pi} s_0} e^{-a^2/2s_0^2}$$

$$p(b) = \frac{1}{\sqrt{2\pi} s_0} e^{-b^2/2s_0^2},$$

where we have dropped the index k since all k-components are identically distributed.

Because the random variables a and b are independent, the joint probability density function

$$\begin{aligned} p_{ab}(a, b) &= p(a)p(b) \\ &= \frac{1}{2\pi s_0^2} e^{-(a^2+b^2)/2s_0^2}. \end{aligned}$$

Now define the spectral power

$$\phi = a^2 + b^2.$$

The probability of observing a power $\leq \phi$ at any particular spectral index is given by the cumulative density function

$$P(\phi) = \int_{a^2+b^2 \leq \phi} p_{ab}(a, b) \mathrm{d}a \mathrm{d}b.$$

Set

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta.$$

Then, integrating over r, from $r^2 = 0$ to $r^2 = \phi$,

$$\begin{aligned} P(\phi) &= \frac{1}{2\pi s_0^2} \int_0^{\sqrt{\phi}} 2\pi r \mathrm{d}r \, e^{-r^2/2s_0^2} \\ &= - \, e^{-r^2/2s_0^2} \Big|_0^{\sqrt{\phi}} \\ &= 1 - e^{-\phi/2s_0^2}. \end{aligned}$$

Recalling that $2s_0^2 = n\sigma^2$, we have

$$P(\phi; n, \sigma^2) = 1 - e^{-\phi/n\sigma^2}.$$

To see what this means, suppose you observe a spectral peak with power ϕ_0 in a time series of length n with a mean-square fluctuation of σ^2.

If the time series were Gaussian white noise with the same mean square fluctuation, the probability of observing a peak with power *greater* than ϕ_0 would be

$$\begin{aligned}\mathrm{Prob}(\phi > \phi_0) &= 1 - P(\phi_0; n, \sigma^2) \\ &= e^{-\phi_0/n\sigma^2}.\end{aligned}$$

This quantity, often called the *p-value*, gives the *statistical significance* of the peak (lower p-values mean greater statistical significance).

As expected, observing a greater power ϕ_0 implies greater statistical significance.

But as the length n of the time series increases, we require a proportionately greater power to maintain the same statistical significance!

Moreover the probability of fluctuations drops off only exponentially with their size.

Thus large fluctuations are common in power spectra, and one must carefully interpret any spectral peak to be confident that it corresponds to a true periodic signal.

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