1.204 Lecture 13

Dynamic programming:
Method
Resource allocation

Introduction

- **Divide and conquer starts with the entire problem, divides it into subproblems and then combines them into a solution**
  - **This is a top-down approach**
- **Dynamic programming starts with the smallest, simplest subproblems and combines them in stages to obtain solutions to larger subproblems until we get the solution to the original problem**
  - **This is a bottom-up approach**
- **Dynamic programming is used much more than divide and conquer**
  - **It is more flexible and controllable**
  - **It is more efficient on most problems since it must consider far fewer combinations**

Principle of optimality

- “Principle of optimality”:
  - In an optimal sequence of decisions or choices, each subsequence must also be optimal
  - For some problems, an optimal sequence may be found by making decisions one at a time and never making a mistake
    - True for greedy algorithms (except label correctors)
  - For many problems it’s not possible to make stepwise decisions based only on local information so that the sequence of decisions is optimal.
    - One way to solve such problems is to enumerate all possible decision sequences and choose the best
    - Dynamic programming can drastically reduce the amount of computation by avoiding sequences that cannot be optimal by the “principle of optimality”

Project selection example

- Suppose we have:
  - $4 million budget
  - 3 possible projects (e.g. flood control)
    - Each funded at $1 million increments from $0 to $4 million
    - Each increment produces a different marginal benefit
  - Dynamic programming problems are usually discrete, not continuous
- We want to find the plan that produces the maximum benefit
- <u>Stages</u> are the number of decisions to be made
  - We have 3 stages, since we have 3 projects
- <u>States</u> are the number of distinct possibilities
  - At each stage there are 5 states ($0, 1, 2, 3, 4 million)

Project selection formulation

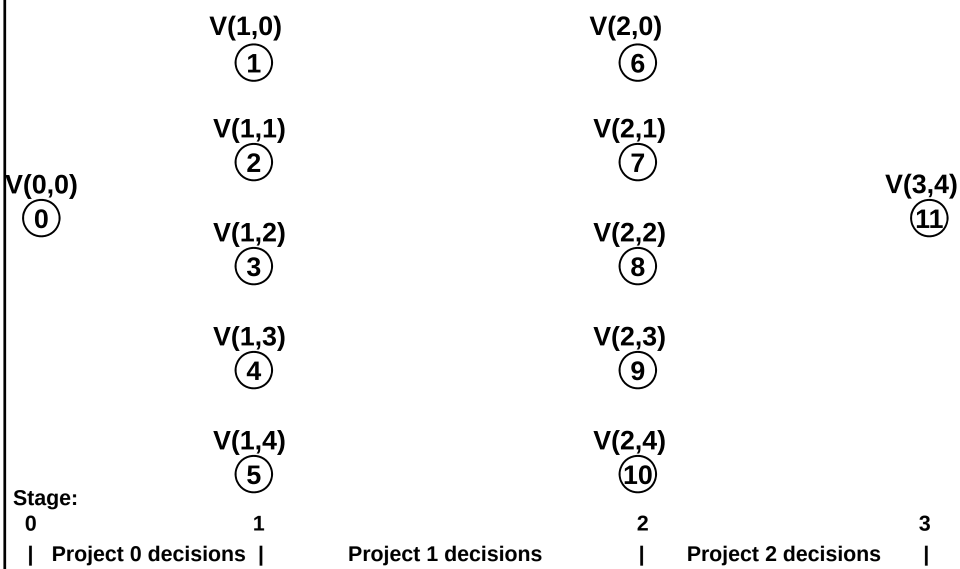
- We build a multistage graph to represent this problem:
  - Source node at start of graph, representing 'null' initial stage
  - Set of nodes at each stage for each state
  - Sink node at end of graph, which is a collapsed representation of the final state
- Each node characterized by V(i,j):
  - V(i,j) is value (benefit) obtained up to (but not including) stage i by committing j resources
  - Each node also stores its predecessor node in P(i)
- Each arc is characterized by E(m,n):
  - E(m,n) is value obtained by spending n resources on project m

Project selection data

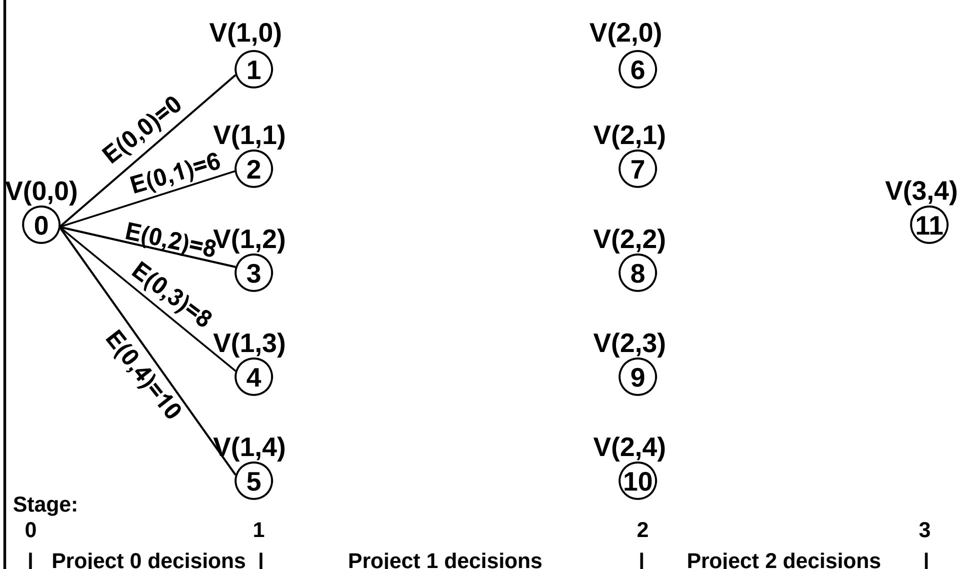
Project 0		Project 1		Project 2	
Investment	Benefit	Investment	Benefit	Investment	Benefit
0	0	0	0	0	0
1	6	1	5	1	1
2	8	2	11	2	4
3	8	3	16	3	5
4	10	4	17	4	6

- In theory, projects could have dependencies, but in practice it's an improbable model. In the example above:
  - Project 1's benefits could depend on project 0 investment
    - But not on project 2 investment
  - Project 2's benefits could depend on total project 0 and 1 investment
    - But not on either individually
- (There are some chip power management graphs with such dependencies)

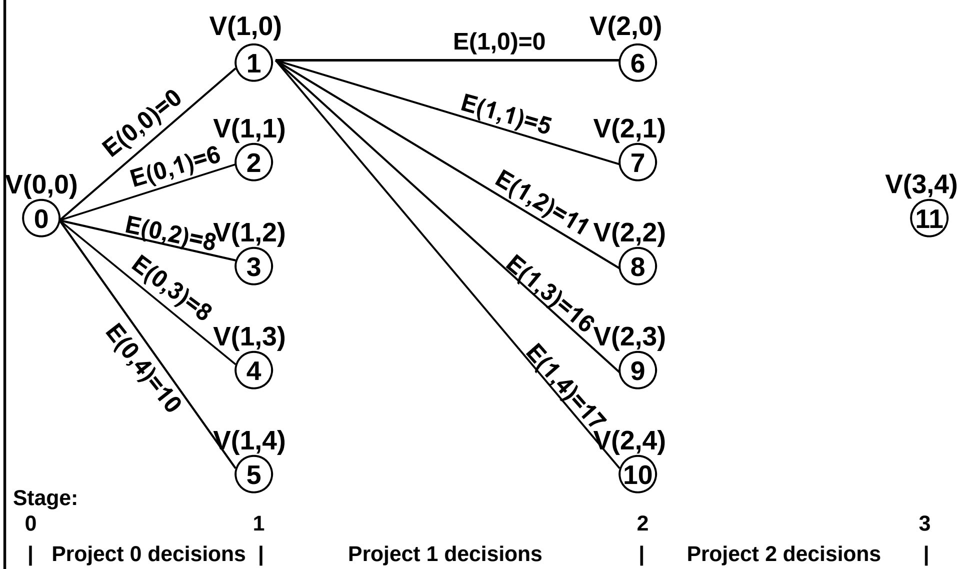
Dynamic programming graph: feasible



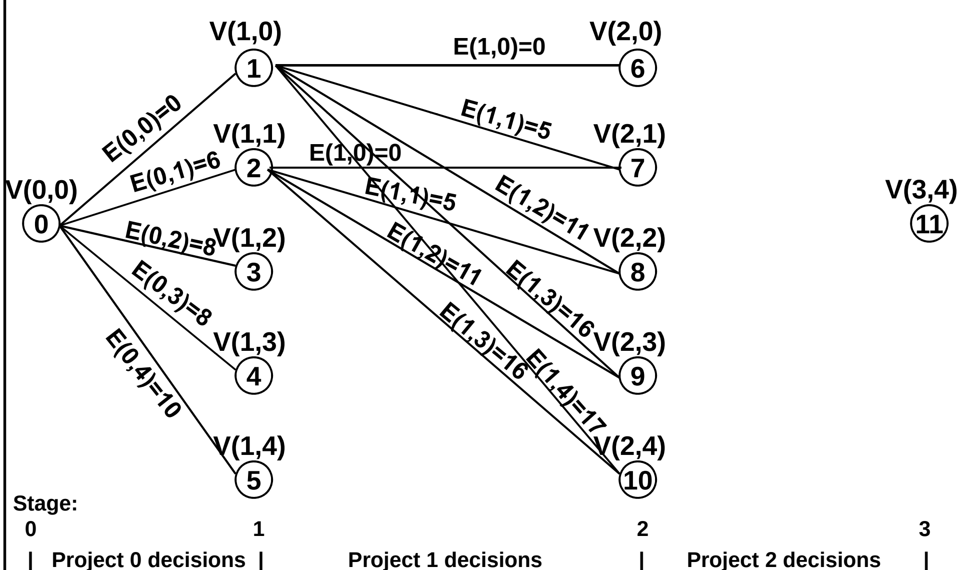
Dynamic programming graph: feasible



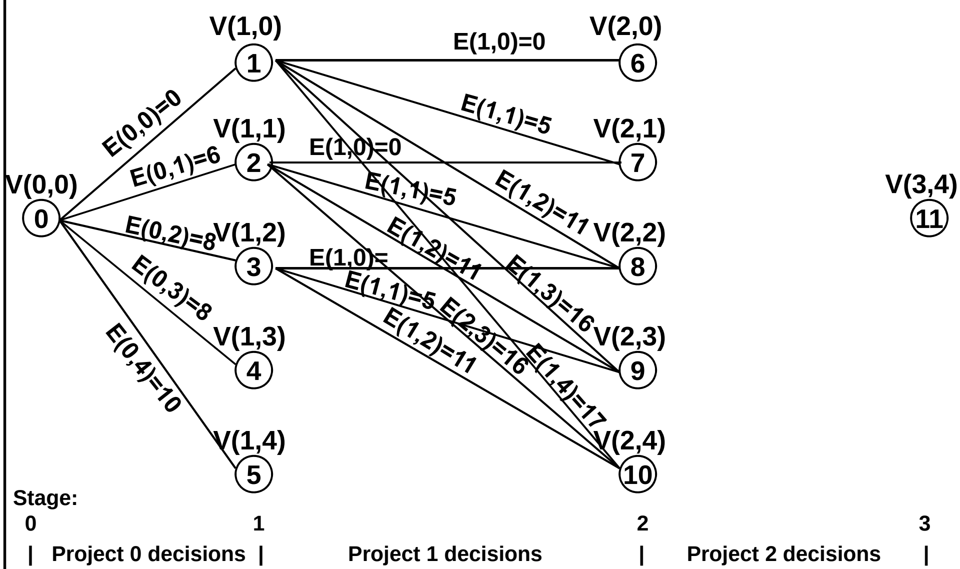
Dynamic programming graph: feasible



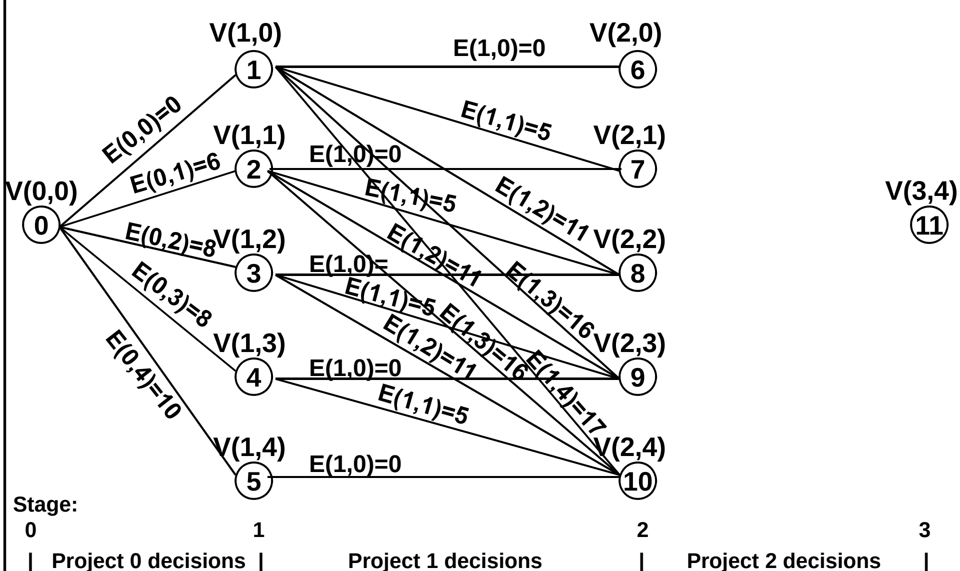
Dynamic programming graph: feasible



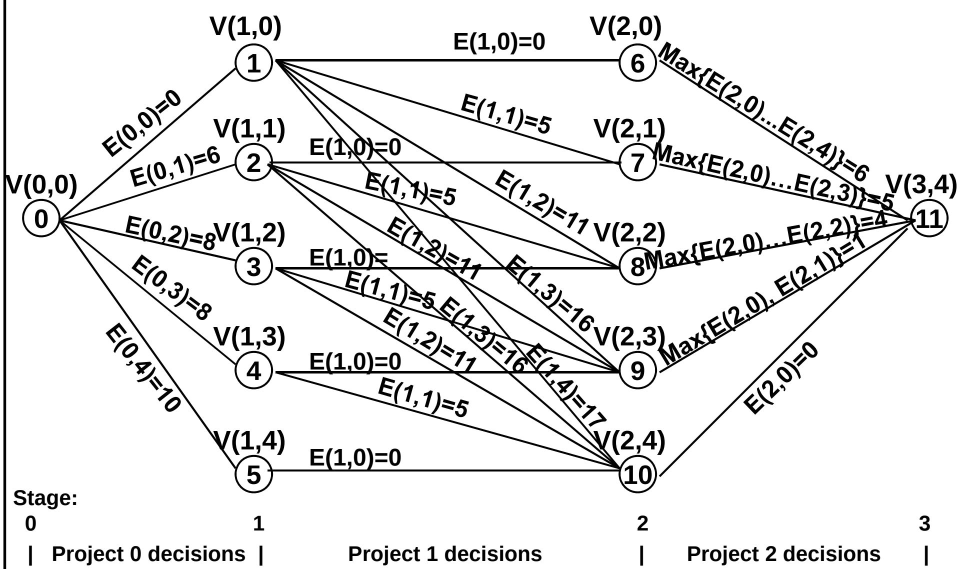
Dynamic programming graph: feasible



Dynamic programming graph: feasible



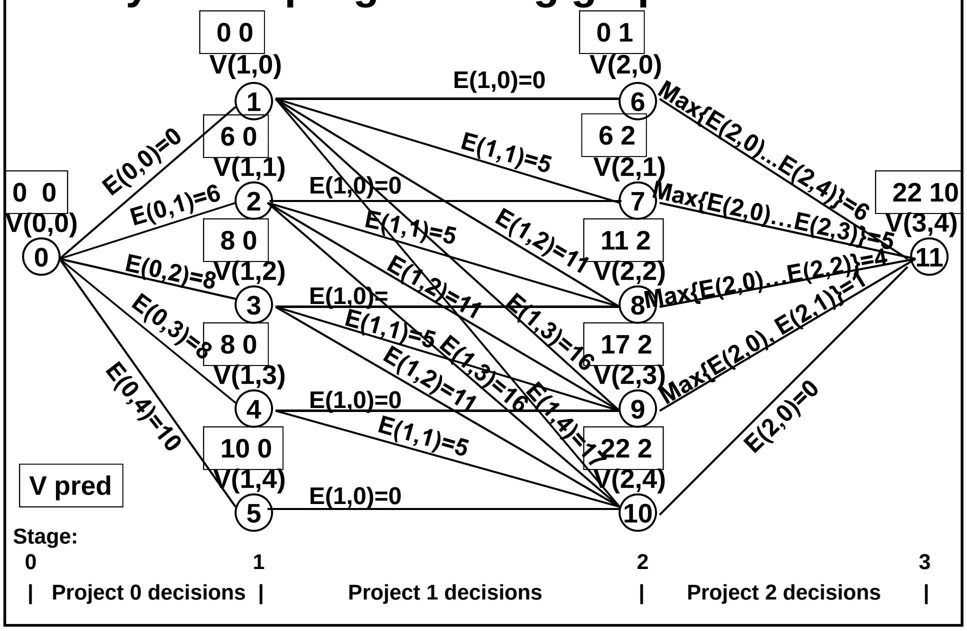
Dynamic programming graph: feasible



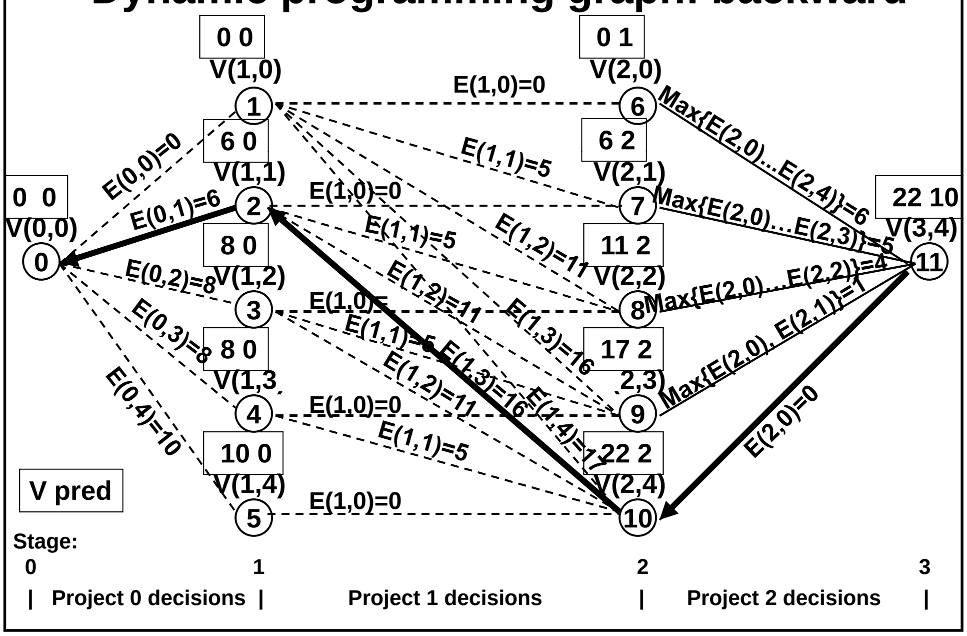
Solution

- **Generate graph in forward direction:**
  - **Start at source node**
  - **Compute V(i,j) and E(m,n) as graph is built**
  - **Keep track of predecessor P(i) of each node that yields highest V(i,j)**
    - **This eliminates non-optimal subsequences (“pruning”)**
  - **Eliminate infeasible arcs and nodes as graph is built**
    - **Rule is easy: Check budget constraint at each node; do not generate arcs or nodes that would violate it**
  - **End when sink node is reached from all nodes of previous stage**
- **Construct solution by tracing back from sink to source using predecessor variable**

Dynamic programming graph: forward



Dynamic programming graph: backward



Multistage graph problem characteristics

- Multistage graph is the standard DP first example
  - Graph is reduced by applying feasibility constraint to eliminate many combinations
    - Can't exceed resource limit
  - Each stage is independent of all previous stages
    - How you got to V(i,j) doesn't matter
    - This limits the combinatorial aspect of the original problem
    - A naïve approach would have looked at all project combinations
- Principle of optimality:
  - "In an optimal sequence of decisions or choices, each subsequence must also be optimal"
  - In our example subsequences are optimal:
    - Node 0 to node 2 (trivially)
    - Node 0 to node 2 to node 10
    - Node 0 to node 2 to node 10 to node 11 (full sequence)

Complexity of multistage graph

- Complexity of well-behaved multistage graph:
  - M projects or stages
  - At each stage, there are ~$n^2/2$ comparisons to find V(i,j) from the incoming arcs
    - Where n is number of resource levels
  - This is $O(Mn^2)$
  - Horowitz and Sahni call it O(M+a)
    - Where a is number of arcs since they assume the graph has already been built and is available as input
- Complexity of worst case:

  Worst case:
    - Different resource levels in each project, so number of nodes increases at each stage
    - High constraint (large resource limit), so no elimination of nodes
    - Number of nodes doubles in each stage
  - This is $O(2^n)$
- Thus, complexity is $O(\min(Mn^2, 2^n))$

Dynamic programming ‘curses’

- Dynamic programming (DP) isn’t natural for most problems
  - Most dynamic programming problems are $O(2^n)$
  - Stages and states have ‘curse of dimensionality’:
    - Stages and states can explode combinatorially
    - Challenge in DP formulation is to avoid or limit the curse…
  - Multistage graph is easiest
  - We’ll do a job scheduling DP next
    - Another example of using the multistage graph model
  - And then it gets harder…
    - We’ll do a set-based DP model for a knapsack problem
    - Sets are “standard model” for complex DP

Multistage graph Java implementation

- Build graph going forward
  - Don’t need graph data structure
    - Don’t need to create or store arcs
    - All information can be stored in nodes
    - Store predecessor of each node (implicit arc)
    - Source, next set of nodes and sink are special cases
- Read off solution going backward from sink
  - Follow predecessors from sink to source
  - Subtract cumulative resources, profits at each step (arc) to recover the decision on each project
- Allocate n+1 nodes per stage if resource limit= n
  - If n= 4, need 5 nodes for resource level 0, 1, 2, 3, 4
- Nested Node class holds profit, resource, predecessor
- Java garbage collector will clean up Nodes not on optimal subsequences
  - No ‘predecessor’ will refer to them

MultistageGraph

```
public class MultiStageGraph {
   private static class Node {
        private int projNbr;      // Project number
        private int cumResource;  // Resource allocated so far
        private int cumProfit;    // Profit so far
        private Node predecessor; // Previous node in graph
        public Node(int proj, int res, int prof, Node p) {
               projNbr= proj;
               cumResource= res;
               cumProfit= prof;
               predecessor= p;
        }
   }

   private int numProj;       // No of projects
   private int n;             // Max units of resource + 1
   private Node root;         // First node in graph
   private Node sink;         // Last node in graph

   public MultiStageGraph(int np, int n) {
        this.numProj = np;
        this.n = n;           // root, sink null initially
   }                          // See download for get, set…
```

MultistageGraph: buildGraph()

```
   public void buildGraph(int[][] p) {  // Profit by project
        // Store previous stage nodes; need at next stage
        Node[] prevStage = new Node[n];
        // Store current stage nodes
        Node[] currStage = new Node[n];
        // Stage 0 start node, units so far 0, profit so far 0
        root = new Node(0, 0, 0, null);

        Node currentNode= null;
        // Project (stage) 1 start nodes as special case,
        // since they have single arcs back to root
        for (int i = 0; i < n; i++) {
               // Stage 1 start node has stage 0 profit
               currentNode = new Node(1, i, p[0][i], root);
               prevStage[i] = currentNode;
        }
```

MultistageGraph: buildGraph() 2

```
// Stage 2 start nodes thru stage M-1 start nodes
for (int i = 2; i < numProj; i++) {
  // Loop, giving 0-> n resources to project
  for (int j = 0; j < n; j++) {
    currentNode = new Node(i, j, 0, null);
    currStage[j] = currentNode;
    for (int k = 0; k <= j; k++) { // Arcs from prev stage
      Node pastNode = prevStage[j - k];
      int profit= p[i-1][k];
      int cumProfit = profit + pastNode.cumProfit;
      if (cumProfit >= currentNode.cumProfit) {
        currentNode.cumProfit=cumProfit;
        currentNode.predecessor=pastNode;
      }
    }
  }
  // Copy current node array into previous node array
  for (int j = 0; j < n; j++) {
    prevStage[j] = currStage[j];
  }
}
```

MultistageGraph: buildGraph() 3

```
// Create the sink, an 'artificial' project M
sink = new Node(numProj + 1, n-1, 0, null);
// Mth project, n units resource, 0 profit
for (int i = 0; i < n; i++) {
  int j= n-1-i;  // Apply max units possible to M-1 project
  Node pastNode = prevStage[i];
  int profit= p[numProj-1][j];
  int cumProfit = profit + pastNode.cumProfit;
    if (cumProfit >= sink.cumProfit) {
      sink.cumResource= j + pastNode.cumResource;
      sink.cumProfit= cumProfit;
      sink.predecessor= pastNode;
    }
  }
  return;
}   // End buildGraph()
```

MultistageGraph: backwardPass()

```
public int backwardPass() {
   System.out.println("Problem solution:");
   System.out.println(" Total profit: " + sink.cumProfit);
   System.out.println(" Total units: " + sink.cumResource+"\n");
   Node next= sink;
   Node current= sink.predecessor;

   while (current != null) {
     System.out.println("Project: "+ current.projNbr);
     // Difference in units is project units assigned
     int units= next.cumResource - current.cumResource;
     // Difference in profit is project profit
     int profit= next.cumProfit - current.cumProfit;
     System.out.println(" Units: "+ units);
     System.out.println(" Profit: "+ profit);
     next= current;
     current= current.predecessor;
   }
   return sink.cumProfit;
}
// Better implementation would return 2-D array of (resource,
// profit) for each project
```

MultistageGraph: main()

```
public static void main(String[] args) {
   int numProjects= 3;
   int maxResource= 4;
   int[][] p2= {{0,6,8,8,10}, // Project 0 profits
        {0,5,11,16,17},       // Project 1 profits
        {0,1,4,5,6},};        // Project 2 profits
   // Increment maxResource: e.g., if maxResource=4,
   // we have 5 decision levels (0,1,2,3,4)
   MultiStageGraph g2=
        new MultiStageGraph(numProjects, ++maxResource);
   g2.buildGraph(p2);
   int totalProfit= g2.backwardPass();
   System.out.println("Total profit: "+ totalProfit);
}
```

Summary

- Dynamic programming key concepts
  - Stages: Decision points
  - States: Decision options
  - Principle of optimality
    - "In an optimal sequence of decisions or choices, each subsequence must also be optimal"
  - Solution approach: create solution graph
    - Eliminate infeasible combinations at each stage
    - Prune suboptimal combinations at each stage
    - Track predecessor of optimal subsequences to each stage
    - (Can generate graph going forward or backward)
  - In most problems, DP is a heuristic solution approach
    - Eliminate/prune unlikely combinations but not provably suboptimal