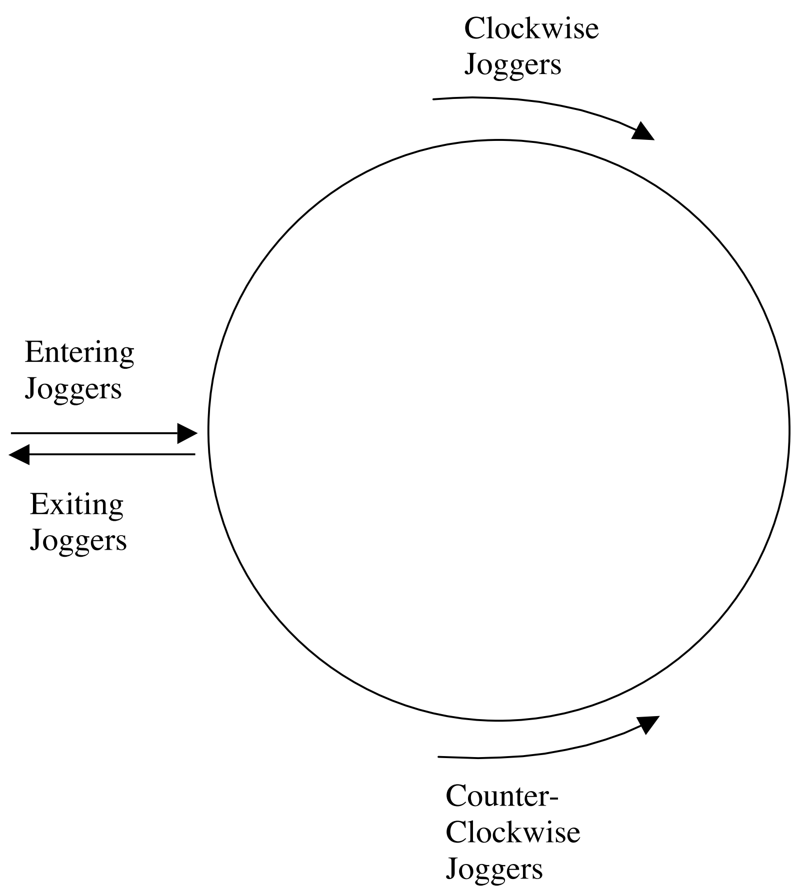
A Jogging Problem.

- Joggers enter the circular jogging loop shown in the figure as a homogeneous Poisson process with rate parameter λ joggers per hour.
- Immediately upon entry to the jogging loop each runner flips a fair coin. Outcomes of flips are mutually independent. If the outcome for a particular jogger is **Heads**, she jogs around the loop in a clockwise manner. If the outcome is **Tails**, she jogs around the loop in a counter-clockwise manner.
- The loop is 1/4 mile in length. All joggers run at the same high speed, running at a rate of 8-minute miles. Thus each completes one loop around in 2 minutes.
- Just before completion of a loop, each jogger again flips a fair coin – while running. If the outcome is **Heads**, she completes her daily run and immediately exits the jogging loop. If the outcome is **Tails**, she continues without any delay (in the same direction) for at least one more loop. This coin flipping process is continued near the completion of each successful loop until the jogger eventually exits the jogging loop.

The Jogging Loop



- Find the mean distance jogged by a random jogger. What is the probability that a jogger will jog more than 3 miles on a given day?

- Assuming the system has been operating for a long time, find the probability that at a random time the entire jogging loop contains *j* joggers, *j* = 0, 1, 2, 3, ….

- Clyde is standing at 12:00 o'clock with a stopwatch. Each time instant that any jogger jogs past him is called a PASS. He is recording the 'gap times' between successive PASSes, regardless of a jogger's direction of travel and regardless of how many times a jogger may have passed him before. Thus the i^{th} gap time that Clyde records is the time (in minutes) between the i+1st PASS and the i^{th} PASS, $i = 1, 2, 3, \ldots$. Suppose he records these inter-PASS gap times for 1,000 joggers and plots those times in a histogram. Please characterize the likely description of this histogram, as to shape and mean value. Do this for
  $\lambda = 500$ joggers per hour and for
  $\lambda = 1.0$ jogger per hour.

SOLUTIONS.

Key is to realize that sums of independent Poisson processes are Poisson. Poisson events in disjoint time intervals are independent. Bernoulli erasers of Poisson events yield Poisson events.

(a) Let L = number of loops that a jogger jogs.
$P\{L=k\} = (1/2)^k$, k=1,2,3,…; E[L] = 2
The mean distance jogged by a random jogger = (1/4)E[L] = (1/4)2 = 0.5 mi.
The probability that a jogger will jog more than 3 miles on a given day =

$$P\{L>12\} = (1/2)^{13} + (1/2)^{14} + (1/2)^{15} + ... = (1/2)^{13}[1 + (1/2) + (1/2)^2 + ...]$$

$$P\{L>12\} = (1/2)^{13}/[1-(1/2)] = (1/2)^{12}$$

We could have written this by inspection, realizing that to jog more than 3 miles requires 12 Tails in a row at end of each of the first 12 loops.

(b) The number of joggers on the track at a random time is equal to the number who arrived within the last 2 minutes (a Poisson r.v. with mean $2\lambda/60$), plus the number who arrived in the previous 2 minutes and who are jogging their 2nd loop (a Poisson r.v. with mean $\lambda/60$), plus the number who arrived in the previous 2 minutes and who are jogging their 3rd loop (a Poisson r.v.

with mean $\lambda/(2*60)$), plus etc., etc. Sums of independent Poisson r.v.'s are Poisson with mean equal to sum of the means. Thus the total number of joggers on the track at a random time is a Poisson r.v., with mean
$(2\lambda+\lambda+\lambda/2+\lambda/4+...)/60 = 2\lambda(1+[1/2]+[1/4]+...)/60 = 4\lambda/60$.
Thus, $P\{J=j\} = (4\lambda/60)^j e^{-4\lambda/60}/j!,\ j=0,1,2,...$

(c) For $\lambda = 500$ joggers per hour Clyde is 'seeing' Poisson passes at a rate of 1000 PASSES per hour [see part (b) above]. 1000 PASSES per hour is 1000/60 = 16.67 PASSES per minute, corresponding to a mean gap of 0.06 minutes or 3.6 seconds. The histogram would resemble a negative exponential curve with mean 3.6 seconds. For $\lambda =$ 1.0 jogger per hour we have two types of PASSES: those of a newly arriving jogger who is on her first loop and repeat joggers who are on other than first loop. This 2nd category is important here as the time between repeats is only 2 minutes (and deterministic) whereas the time between new 1st loop joggers is one hour and is exponentially distributed. Roughly half the PASSES will be joggers on their first loop and the other half will be joggers on 2nd or later loops. Thus approximately half of the inter-PASS gaps will be negative exponential with mean one hour and the remainder will be deterministic at 2 minutes. The histogram would thus resemble a negative exponential curve with mean one hour with an added spike or impulse at 2 minutes, with approximately 50% of the area under the total curve.