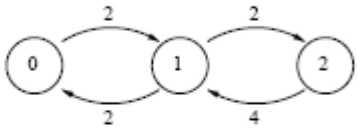
Problem Set #5

Problem 1

(a) One is tempted to say yes by setting $\rho = \frac{\lambda}{N\mu} = \frac{2}{2\times 2} = \frac{1}{2}$. But $\lambda = 2$ is not the rate at which customers are accepted into the system because we have a loss system. Thus the answer is no, and we must derive the correct figure. We can use the following aggregate birth-death process (state transition diagram for an M/M/2 queueing system with no waiting space) to compute the workloads:



The balance equations and the normalization equation are

$$\begin{aligned} 2P_0 &= 2P_1 \\ 2P_1 &= 4P_2 \\ P_0 + P_1 + P_2 &= 1 \end{aligned}$$

Solving the equations, we obtain

$$P_0 = \frac{2}{5}, \quad P_1 = \frac{2}{5}, \quad P_2 = \frac{1}{5}.$$

The workloads of server 1 and server 2 are then given by

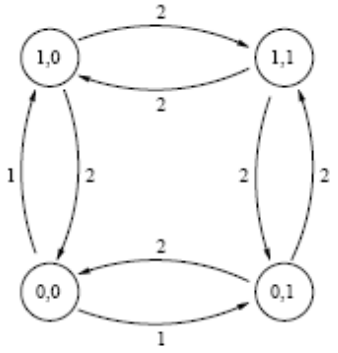
$$\rho_1 = \frac{1}{2}P_1 + P_2 = \frac{2}{5}, \quad \rho_2 = \frac{1}{2}P_1 + P_2 = \frac{2}{5}.$$

(b) The 2-dimensional hypercube state transition diagram is given below. From the steady-state probabilities computed in part (a) and the symmetry of the system, we have

$$P_{00} = P_0 = \frac{2}{5}, \quad P_{11} = P_2 = \frac{1}{5}, \quad P_{10} = P_{01} = \frac{1}{2}P_1 = \frac{1}{5}.$$

The fraction of dispatches that take server 1 to sector 2 is

$$f_{12} = \frac{\lambda_2}{(1 - P_{11})\lambda} P_{10} = \frac{1}{(1 - \frac{1}{5})2}\left(\frac{1}{5}\right) = \frac{1}{8}.$$



(c) The mean travel time to a random served customer, $\bar{T}$, is obtained by

$$\bar{T} = f_{11}\,T_1(\text{sector 1}) + f_{22}\,T_2(\text{sector 2}) + f_{12}\,T_1(\text{sector 2}) + f_{21}\,T_2(\text{sector 1})\,.$$

Since the travel speed is constant, let us first compute the mean travel distance to a random customer, $\bar{D}$.

$$\bar{D} = f_{11}\,D_1(\text{sector 1}) + f_{22}\,D_2(\text{sector 2}) + f_{12}\,D_1(\text{sector 2}) + f_{21}\,D_2(\text{sector 1})\,.$$

Using the knowledge of Chapter 3, we have

$$D_1(\text{sector 1}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}, \quad D_2(\text{sector 2}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3},$$
$$D_1(\text{sector 2}) = 1 + \frac{1}{3} = \frac{4}{3}, \quad D_2(\text{sector 1}) = 1 + \frac{1}{3} = \frac{4}{3}.$$

We compute f_{11} as follows:

$$f_{11} = \frac{\lambda_1}{(1 - P_{11})\lambda}(P_{00} + P_{10}) = \frac{1}{(1 - \frac{1}{5})2}\left(\frac{2}{5} + \frac{1}{5}\right) = \frac{3}{8}\,.$$

Invoking the symmetries, we know

$$f_{21} = f_{12} = \frac{1}{8}, \quad f_{22} = f_{11} = \frac{3}{8}\,.$$

Putting all together,

$$\bar{D} = \frac{3}{8}\cdot\frac{2}{3} + \frac{3}{8}\cdot\frac{2}{3} + \frac{1}{8}\cdot\frac{4}{3} + \frac{1}{8}\cdot\frac{4}{3} = \frac{5}{6}\ \text{mile}.$$

Hence the mean travel time to a random served customer is $\bar{T} = \frac{\bar{D}}{1000}$ hr $= 3.0$ sec. This

means that changes in total service time due to changes in travel time are insignificant and therefore the Markov models applies. Note that another way to compute $\bar{D}$ is

$$\bar{D} = \frac{P_{00}(\frac{2}{3}) + (P_{01} + P_{10})(\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{4}{3})}{P_{00} + P_{01} + P_{10}} = \frac{\frac{2}{5}(\frac{2}{3}) + \frac{2}{5}(\frac{1}{2} \cdot \frac{4}{3} + \frac{1}{2} \cdot \frac{2}{3})}{\frac{4}{5}} = \frac{5}{6}.$$

In fact, we can obtain this form by simplifying the formula for $\bar{D}$ above. However, think about how we can obtain this directly without using the formula for $\bar{D}$ above.

(d) Consider a long time interval T. In the steady state, the average total number of customers served is $\lambda T(1 - P_{11})$. Server 1 is sent to sector 2 in the following cases:

(1) A customer arrives from sector 2, server 2 is busy, and server 1 is idle.

(2) A customer arrives from buffer zone 2, server 2 is idle outside buffer zone 2, and server 1 is idle inside buffer zone 1.

The average number of customers served by the first case is $\lambda_2 T P_{10}$. To compute the average number of customers served by the second case, let us first find the probability that server 2 is idle *outside* buffer zone 2 and server 1 is idle *inside* buffer zone 1. Using geometrical probability and the independence of the two servers, we know that the probability is $(\frac{3}{4})(\frac{1}{4})P_{00}$. Since the arrival rate from buffer zone 2 is $\frac{\lambda_2}{4}$, the average number of customers served by the second case during time interval T is $\frac{\lambda_2}{4}T(\frac{3}{4})(\frac{1}{4})P_{00}$.

Using these quantities, we obtain the fraction of dispatch assignments that send server 1 to sector 2 under the new dispatch policy as follows:

$$f'_{12} = \frac{\lambda_2 T P_{10} + \frac{\lambda_2}{4}T(\frac{3}{4})(\frac{1}{4})P_{00}}{\lambda T(1 - P_{11})} = \frac{1 \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{2}{5}}{2(1 - \frac{1}{5})} = \frac{35}{256} = 0.1367.$$

f'_{12} is greater than $f_{12} = 0.125$ as expected. Note that the state transition diagram does not change under the new dispatch policy (Why? Invoke symmetries).

(e) Let T_1 be the travel time of server 1 to a random customer and T_2 be the travel time of server 2 to a random customer. Similar to (c), the mean travel time to a random customer under the new dispatch policy is given by

$$\begin{aligned}\bar{T}' = & f'_{11}\, E[T_1 \mid \text{server 1 has been dispatched into sector 1}] + \\ & f'_{22}\, E[T_2 \mid \text{server 2 has been dispatched into sector 2}] + \\ & f'_{12}\, E[T_1 \mid \text{server 1 has been dispatched into sector 2}] + \\ & f'_{21}\, E[T_2 \mid \text{server 2 has been dispatched into sector 1}].\end{aligned}$$

But the existence of buffer zones complicates matters. One way to handle this is as follows:

- Break up f'_{12} (and f'_{21}) into its two constituent parts and compute a conditional mean travel distance for each
- Do the same for f'_{11} and f'_{22}.
- Combine the results for the final answer.

The numerical value is less than that of part (c), because we tend to dispatch the closer available server (not always successful, though).

Although it is not required in the question, let us compute T' exactly. We define the following events:

- CB: A customer is in a buffer zone.
- SAB: Server of the adjacent sector is in its buffer zone.
- SHB: Server of home sector is in its buffer zone.

Let us denote by CB^c the complement event of CB, which means that a customer is not in a buffer zone. Other complement events are defined similarly. Then in the state where both servers are available, with probability P_{00}, we have eight mutually exclusive, collective exhaustive events: $(\mathrm{CB}\cap\mathrm{SAB}\cap\mathrm{SHB})$, $(\mathrm{CB}^c\cap\mathrm{SAB}\cap\mathrm{SHB})$, $(\mathrm{CB}\cap\mathrm{SAB}^c\cap\mathrm{SHB})$, $(\mathrm{CB}\cap\mathrm{SAB}\cap\mathrm{SHB}^c)$, $(\mathrm{CB}^c\cap\mathrm{SAB}^c\cap\mathrm{SHB})$, $(\mathrm{CB}^c\cap\mathrm{SAB}\cap\mathrm{SHB}^c)$, $(\mathrm{CB}\cap\mathrm{SAB}^c\cap\mathrm{SHB}^c)$, and $(\mathrm{CB}^c\cap\mathrm{SAB}^c\cap\mathrm{SHB}^c)$.

Let us abbreviate these events in binary, for example, $(\mathrm{CB}\cap\mathrm{SAB}\cap\mathrm{SHB}) = (111)$, $(\mathrm{CB}^c\cap\mathrm{SAB}\cap\mathrm{SHB}^c) = (010)$, etc. Then we can write, using the techniques from Chapter 3 for the conditional mean travel distances,

$$D' = \frac{P_{00}A + (P_{01}+P_{10})(\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{4}{3})}{P_{00}+P_{01}+P_{10}},$$

where A is

$$\begin{aligned} A = &\left(\frac{1}{3}+\frac{1}{4}\right)P(110) + \left[\frac{1}{3}+\left(\frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{4}\right)\right]P(100) + \\ &\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{4}\right)\left(P(101)+P(111)\right) + \left(\frac{1}{3}+\frac{1}{3}\cdot\frac{3}{4}\right)\left(P(010)+P(000)\right) + \\ &\left[\frac{1}{3}+\left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\right)\right]\left(P(001)+P(011)\right). \end{aligned}$$

We have :

$$P(110) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4}, \quad P(100) = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}, \quad P(101) = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}, \quad P(111) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4},$$
$$P(010) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4}, \quad P(000) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}, \quad P(001) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}, \quad P(011) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}.$$

Plugging all numbers, we obtain $\bar{D}' = \frac{1271}{1536} = 0.82747 < \bar{D} = \frac{5}{6} = 0.83333$. The mean travel time to a random customer is $\bar{T}' = \frac{\bar{D}'}{1000} = 2.9789$ sec $< \bar{T} = 3$ sec. So, we do get an expected improvement in mean response distance (time), but not a large one. The fact that we have more inter-sector dispatches *does not* necessarily mean that mean response distance (time) will increase.

(f) First, do not use Carter, Chaiken, and Ignall formula (Equation (5.18)). It only applies when server locations are fixed. The best option is to compute $\bar{T}(x)$, where x is the location of a boundary line, and use calculus to find an optimal value of x (as we did in the 2-server numerical example in the book and in class). The problem with Equation (5.18) is that $T_1(B)$ and $T_2(B)$ depend on the location of the boundary line separating sectors 1 and 2. This is because each available server patrols uniformly its sector while it is idle and thus its travel time in B depends on sector design.

Problem 2

(a) With probability 0.3, the emergency occurs on one of the two links incident to the garage location of ambulance 2. In this case, the travel distance is U(0, 1). With probability 0.7, the emergency occurs on one of the two links not incident to the garage location of ambulance 2. In this case, the travel distance is U(1, 2). Accordingly, as shown in Figure 1, the conditional pdf of the travel distance for ambulance 2 to travel to the scene of the emergency incident is

$$f_D(d) = \begin{cases} 0.3, & d \in [0,1) \\ 0.7, & d \in [1,2] \end{cases}$$

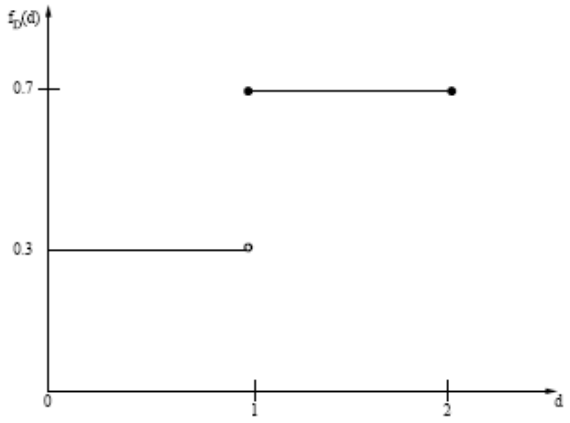


Figure 1: Conditional pdf of Travel Distance

(b)

The state transition diagram for this system is shown in Figure 2, where first component of the state indicates whether ambulance 2 is busy and the second component indicates whether ambulance 1 is busy. We can thus write the following balance equations.

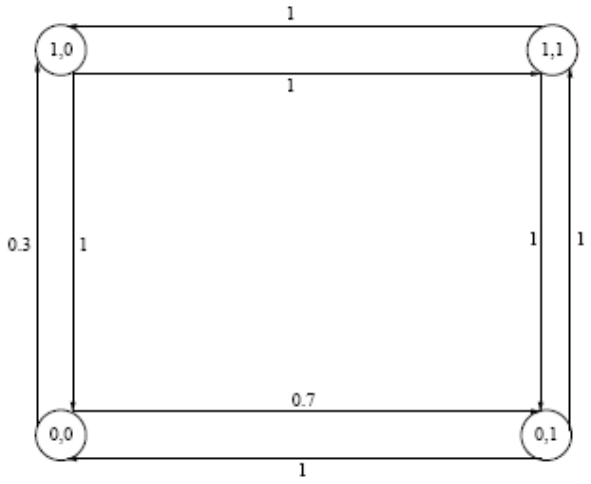


Figure 2: State Transition Diagram

$$\begin{aligned} P_{0,0}(0.3+0.7) &= P_{0,1}+P_{1,0} \\ 2P_{0,1} &= 0.7P_{0,0}+P_{1,1} \\ 2P_{1,1} &= P_{1,0}+P_{0,1} \\ P_{0,0}+P_{0,1}+P_{1,0}+P_{1,1} &= 1 \end{aligned}$$

Solving this system, we obtain

$$\begin{aligned} P_{0,0} &= \frac{2}{5} \\ P_{0,1} &= \frac{6}{25} \\ P_{1,0} &= \frac{4}{25} \\ P_{1,1} &= \frac{1}{5} \end{aligned}$$

Therefore,

$$\begin{aligned} \rho_1 &= P_{0,1}+P_{1,1} = \frac{11}{25} \\ \rho_2 &= P_{1,0}+P_{1,1} = \frac{9}{25} \end{aligned}$$

(c) This is a straightforward application of Equation 5.18 from the textbook.

$$s_0 = \frac{2\eta}{2\eta+1}\left(T_2(B)-T_1(B)\right)$$

where $T_n(B)$ gives the average travel time for unit n to travel to a random service request from anywhere in the entire service region. Note that s_0 is therefore given in time units. Let us multiply the RHS through by the travel speed and let $D_n(B)$ denote the average travel distance for unit n to travel to a random service request from anywhere in the entire service region. We can then write s_0 as follows, in units of distance rather than time.

$$\begin{aligned} s_0 &= \frac{2\eta}{2\eta+1}\left(D_2(B) - D_1(B)\right) \\ D_1(B) &= 0.1 \cdot 1.5 + 0.2 \cdot 1.5 + 0.3 \cdot 0.5 + 0.4 \cdot 0.5 = 0.8 \\ D_2(B) &= 0.1 \cdot 0.5 + 0.2 \cdot 0.5 + 0.3 \cdot 1.5 + 0.4 \cdot 1.5 = 1.2 \\ \eta &= \frac{\lambda}{2\mu} = \frac{1}{2} \\ s_0 &= \frac{0.4}{2} = 0.2 \text{ km} \end{aligned}$$

This means we shift the equal-travel-time boundary line away from the northwest and southeast corners of the square and toward the northeast corner of the square, but moving only 0.2 km in those directions.

Problem 3

Consider a point x on the circumference. In steady-state,

$$\begin{aligned} \mathrm{P}(x \text{ covered}|\text{one car busy}) &= \frac{2a}{2\pi M} = \frac{a}{\pi M} \\ \mathrm{P}(x \text{ covered}|N \text{ cars busy}) &= 1 - \mathrm{P}(x \text{ not covered by any of the } N \text{ cars}) \\ &= 1 - (1 - \frac{a}{\pi M})^N \end{aligned}$$

To find the unconditional probability, P(x covered), we must find out P(N cars busy).

Note that the busy cars form an M/G/∞ case since they have Poisson arrivals, general service time pdf and there are enough cars to answer any number of calls. Therefore,

$$\mathrm{P}(N \text{ cars busy}) = \mathrm{P}_N \text{ for M/G/}\infty = \frac{(\frac{\lambda'}{\mu})^N e^{-\frac{\lambda'}{\mu}}}{N!}$$

where $\lambda' = 2\pi M\lambda$.

Hence,

$$\begin{aligned} \mathrm{P}(x \text{ covered}) &= \sum_{N=0}^{\infty} \left\{ \left[1 - (1 - \frac{a}{\pi M})^N\right] \frac{(\frac{\lambda'}{\mu})^N e^{-\frac{\lambda'}{\mu}}}{N!} \right\} \\ &= e^{-\frac{\lambda'}{\mu}} \left\{ \sum_{N=0}^{\infty} \frac{(\frac{\lambda'}{\mu})^N}{N!} - \sum_{N=0}^{\infty} \frac{[(1 - \frac{a}{\pi N})\frac{\lambda'}{\mu}]^N}{N!} \right\} \\ &= e^{-\frac{\lambda'}{\mu}} \left[e^{\frac{\lambda'}{\mu}} - e^{\frac{\lambda'}{\mu}(1 - \frac{1}{\pi N})} \right] \\ &= 1 - e^{-\frac{\lambda'}{\mu}(\frac{a}{\pi N})} \\ \Rightarrow \quad \mathrm{P}(x \text{ covered}) &= 1 - e^{-\frac{2a\lambda}{\mu}} \end{aligned}$$

Using the argument of dividing the total length $2\pi M$ into infinitesimal small intervals,

$$\mathrm{E}(\text{length covered}) = 2\pi M(1 - e^{-\frac{2a\lambda}{\mu}})$$

alternative solution Since no point is specially favored in this problem, expected length that being covered in steady state should be equal total length times steady state of a point being covered by any one of the cars, i.e.

$$\mathrm{E}(\text{length covered}) = 2\pi M \times \mathrm{P}(\text{point } x \text{ is covered})$$

We can apply M/G/∞ model with parameter $2a(\frac{\lambda}{\mu})$ to calculate P(x is covered). Hence,

$$\mathrm{P}(x \text{ is covered}) = 1 - P_0 = 1 - e^{-2a(\frac{\lambda}{\mu})}$$

Thus, E(length covered) $= 2\pi M \times (1 - e^{-2a(\frac{\lambda}{\mu})})$.

Problem 4

(a) Let us denote the state of the system by a 3-character list of the status of each server (0 indicates free and 1 indicates busy), where the status of server 1 is given by the right-most character and that of server 3 is given by the left-most character. Let P_n denote the probability that a total of n emergencies are in the system. By the symmetry of the problem, it is clear that

$$P_{100} = P_{010} = P_{001} = \frac{1}{3}P_1$$
$$P_{110} = P_{101} = P_{011} = \frac{1}{3}P_2$$

Since all mean service times are equal, P_n can be found by looking at the equivalent $M/M/3$ system with $\lambda = 2$ and $\mu = 1$. (See pp. 305-07 of the textbook for a discussion). From equations (4.46) and (4.44) in the Urban OR textbook, we obtain that

$$\begin{aligned} P_0 &= \frac{1}{1+2+2+\frac{4}{3(1-\frac{2}{3})}} \\ &= \frac{1}{1+2+2+4} = \frac{1}{9} \\ P_1 = P_2 &= \frac{2}{9} \\ P_n &= \frac{2^n}{54 \cdot 3^{n-3}},\ n \in \{3,4,5,\dots\} \end{aligned}$$

Furthermore, by the definition of the states, $i \in \{1,2,3\}$, we have

$$\begin{aligned} \rho_i &= \frac{1}{3}P_1 + \frac{2}{3}P_2 + \sum_{j=3}^{\infty} P_j \\ &= \frac{2}{27} + \frac{4}{27} + (1 - P_0 - P_1 - P_2) \\ &= \frac{2}{3} \end{aligned}$$

(b) Without loss of generality (by the symmetry of the problem), we can analyze the mean travel time by assuming that the call originates from district 1. We will use the Total Expectation Theorem by conditioning on the following mutually exclusive and exhaustive events

- A: unit 1 is available when the call arrives. $P(A) = 1 - \rho_1 = \frac{1}{3}$.
- B: unit 1 is unavailable when the call arrives but at least one of units 2 or 3 is available. $P(B) = P_{001} + P_{101} + P_{011} = \frac{2}{9}$
- C: no unit is available when the call arrives. $P(C) = 1 - P_0 - P_1 - P_2 = \frac{4}{9}$.

Recall that if unit 1 is unavailable but units 2 and 3 are available, then one of 2 or 3 is dispatched to the call. In either case, the expected travel time is 2 minutes. Now consider case C. Due to the memorylessness of the exponential distribution and the fact that the service time distributions are the same for each unit, the call is equally likely to be served by any of the three units. So, the expected travel time in this case is $\frac{1}{3}\left(\frac{1}{2}+2+2\right) = \frac{3}{2}$. Putting this all together, we have that

$$\begin{aligned} E[T] &= E[T \mid A]P(A) + E[T \mid B]P(B) + E[T \mid C]P(C) \\ &= \frac{1}{2} \cdot \frac{1}{3} + 2\frac{2}{9} + \frac{3}{2} \cdot \frac{4}{9} \\ &= \frac{23}{18} \approx 1.28 \text{ minutes} \end{aligned}$$

(c) Again, we can condition on events A, B, and C. For event A, $T \sim \mathrm{U}(0,1)$. For event B, $T \sim \mathrm{U}(1,3)$. For event C, $T \sim \mathrm{U}(0,1)$ w.p. $\frac{1}{3}$ (unit 1 becomes free before 2 or 3) and $T \sim \mathrm{U}(1,3)$ w.p. $\frac{2}{3}$ (one of units 2 or 3 becomes free before unit 1). That is

$$
\begin{aligned}
f_{T|A}(t) &= \begin{cases} 1, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases} \\
f_{T|B}(t) &= \begin{cases} \frac{1}{2}, & t \in [1,3] \\ 0, & \text{otherwise} \end{cases} \\
f_{T|C}(t) &= \begin{cases} \frac{1}{3}, & t \in [0,3] \\ 0, & \text{otherwise} \end{cases} \\
f_T(t) &= f_{T|A}(t)P(A) + f_{T|B}(t)P(B) + f_{T|C}(t)P(C) \\
&= \begin{cases} \frac{13}{27}, & t \in [0,1] \\ \frac{7}{27}, & t \in [1,3] \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

(d) Let x denote a location on the circle, where $x \in [0,6)$. Let $f(x)$ be a random indicator function of a point x where

$$
f(x) = \begin{cases} 1, & x \text{ covered} \\ 0, & \text{otherwise} \end{cases}
$$

Of course, the value of $f(x)$ will depend on the random locations of the units. The total random amount of the city that is covered at a random time is given by $\int_0^6 f(x)dx$. The average (i.e. expected) amount of the city that is covered at a random time is $E\left[\int_0^6 f(x)dx\right]$. Recall that the expected value of a sum of random variables is always the sum of the expected values, regardless of whether the random variables are independent or not (linearity of expectation). Since the integration inside the expectation is essentially the same as summation (we're just summing over tiny intervals), we have that

$$
\begin{aligned}
E\left[\int_0^6 f(x)dx\right] &= \int_0^6 E[f(x)]dx \\
&= \int_0^6 P(x \text{ covered})dx
\end{aligned}
$$

where the last equality follows from the fact that the expected value of an indicator function is equal to the probability that the indicator equals 1. Now, it is easy to see that $P(x \text{ covered})$ may not be the same for all values of $x \in [0,6)$. for instance, if x equals the home location of one of the units, then the probability of coverage is likely to be higher. However, for certain intervals, it turns out that $P(x \text{ covered})$ is constant. In particular, we can break our analysis into two cases.

First, let us derive γ_{ij}, the fractions of i dispatches whose destination is district j, $i, j \in \{1, 2, 3\}$. Note that the γ_{ij} are not the same as the f_{ij} defined in the text, since the latter is given to be the fraction of *all* dispatches (not just i dispatches), which take unit i to district j. We will derive the γ_{ij} from the f_{ij} via renormalization. Recall that each f_{ij} is the sum of a term corresponding to the fraction of dispatches of i to j that incur no queueing delay and those that do incur a delay in queue. For any call delayed in queue (arrives when all of the units are busy), because of the memorylessness of the exponential distribution, the call is equally likely to be answered by any one of the three units.

$$\begin{aligned}
f_{11} &= P(\text{call arr to 1})\,(P(1 \text{ free}) + P(\text{all busy, and 1 becomes free first})) \\
&= \frac{1}{3}\left(1 - \rho_1 + \frac{1}{3}(1 - P_0 - P_1 - P_2)\right) \\
&= \frac{1}{3}\left(\frac{1}{3} + \frac{1}{3}\cdot\frac{4}{9}\right) = \frac{13}{81} \\
f_{ii} &= \frac{13}{81}, \forall i \in \{1, 2, 3\}, \text{ by symmetry} \\
f_{12} &= P(\text{call arr to 2})\left(P_{110} + \frac{1}{2}P_{010} + \frac{1}{3}(1 - P_0 - P_1 - P_2)\right) \\
&= \frac{1}{3}\left(\frac{2}{27} + \frac{1}{2}\cdot\frac{2}{27} + \frac{1}{3}\cdot\frac{4}{9}\right) \\
&= \frac{7}{81} \\
f_{ij} &= \frac{7}{81}, \ \forall i \neq j, \text{ by symmetry}
\end{aligned}$$

So, to derive the γ_{ij}, we must renormalize so that the sample space includes only dispatches of unit i. Again, we can exploit the symmetry of the problem to conclude that

$$\begin{aligned}
\gamma_{ii} &= \frac{f_{11}}{f_{11} + f_{12} + f_{13}} = \frac{13}{27}, \ \forall i \in \{1, 2, 3\} \\
\gamma_{ij} &= \frac{f_{12}}{f_{11} + f_{12} + f_{13}} = \frac{7}{27}, \ \forall i \neq j
\end{aligned}$$

Now, we are ready to derive $P(x \text{ covered})$. By the symmetry of the problem, we derive this result by assuming, without loss of generality, that x is located within district 1. Now, we need to consider the following two cases. Let E_i denote the event that x is covered by unit i.

- **Case 1:** x is within $\frac{1}{2}$ mile of the home location of unit 1.

To explain the entries in the following table, note that a unit located within district 1 covers x if it is in the $\frac{1}{2}$ mile to the right or to the left of x. Thus, the unit must be in one of two subintervals whose lengths add to 1 mile. Given that a unit is in district 1, its exact location within the district is uniformly distributed. Therefore, given that a unit is in district 1, the probability that it covers x is simply $\frac{1}{2}$ (since district 1 has total length 2 miles). Let Q denote the state in which the number n of emergency calls in the system is at least 4.

States	$\Pr(E_1)$	$\Pr(E_2)$	$\Pr(E_3)$	$\Pr(x \text{ covered})$
000, 010, 100, 110	1	who cares	who cares	1
001	$\frac{\gamma_{11}}{2}$	0	0	$\frac{\gamma_{11}}{2}$
011	$\frac{\gamma_{11}}{2}$	$\frac{\gamma_{21}}{2}$	0	$1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{21}}{2}\right)$
101	$\frac{\gamma_{11}}{2}$	0	$\frac{\gamma_{31}}{2}$	$1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{31}}{2}\right)$
111, Q	$\frac{\gamma_{11}}{2}$	$\frac{\gamma_{21}}{2}$	$\frac{\gamma_{31}}{2}$	$1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{21}}{2}\right)\left(1-\frac{\gamma_{31}}{2}\right)$

The probability that x is covered, given Case 1, is

$$\begin{aligned} P(x \text{ covered} \mid \text{Case 1}) &= (1-\rho_1) + P_{001}\frac{\gamma_{11}}{2} \\ &\quad + P_{011}\left[1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{21}}{2}\right)\right] \\ &\quad + P_{101}\left[1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{31}}{2}\right)\right] \\ &\quad + (P_{111}+P_Q)\left[1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{21}}{2}\right)\left(1-\frac{\gamma_{31}}{2}\right)\right] \\ &= 0.5902 \end{aligned}$$

- **Case 2:** x is in district 1 but further than $\frac{1}{2}$ mile from the home location of unit 1.

  Without loss of generality, we can assume that x is located no more than $\frac{1}{2}$ mile from the district 1-2 boundary (otherwise, the analysis is the same but involves districts 1 and 3, rather tan districts 1 and 2). Suppose that x is located in district 1, D units from the district 1-2 boundary. $D \sim \mathrm{U}\left(0, \frac{1}{2}\right)$.

  Suppose we are given that $D = d$. A unit in district 1 covers x iff it lies in the d units between x and the district 1-2 boundary or lies in the $\frac{1}{2}$ mile on the other side of x. Thus, given that a unit is in district 1, and given d, the probability that it covers x is $\frac{\frac{1}{2}+d}{2}$, since district 1 has total length 2 miles. In contrast to Case 1, we now also have that a unit in district 2 may cover x, even though x is in district 1. In particular, a unit in district 2 covers x iff it is in the $\frac{1}{2}-d$ mile of district 2 that borders on the district 1-2 boundary. So, given that a unit is in district 2, and given d, the probability that it covers x is $\frac{\frac{1}{2}-d}{2}$.

  However, the probabilities that we gave above are conditional probabilities, where we've conditioned on the distance d of x from the district 1-2 boundary. The corresponding unconditional probabilities are as follows. Given that a unit is in district 1, the probability that it covers x is $\int_0^{\frac{1}{2}} 2 \cdot \frac{\frac{1}{2}+u}{2} du = \frac{3}{8}$. Similarly, given that a unit is in district 2, the probability that it covers x is $\int_0^{\frac{1}{2}} 2 \cdot \frac{\frac{1}{2}-u}{2} du = \frac{1}{8}$.

States	$\Pr(E_1)$	$\Pr(E_2)$	$\Pr(E_3)$
000	0	0	0
001	$\frac{3}{8}\gamma_{11}+\frac{1}{8}\gamma_{12}$	0	0
010	0	$\frac{3}{8}\gamma_{21}+\frac{1}{8}\gamma_{22}$	0
100	0	0	$\frac{3}{8}\gamma_{31}+\frac{1}{8}\gamma_{32}$
011	$\frac{3}{8}\gamma_{11}+\frac{1}{8}\gamma_{12}$	$\frac{3}{8}\gamma_{21}+\frac{1}{8}\gamma_{22}$	0
101	$\frac{3}{8}\gamma_{11}+\frac{1}{8}\gamma_{12}$	0	$\frac{3}{8}\gamma_{31}+\frac{1}{8}\gamma_{32}$
110	0	$\frac{3}{8}\gamma_{21}+\frac{1}{8}\gamma_{22}$	$\frac{3}{8}\gamma_{31}+\frac{1}{8}\gamma_{32}$
111, Q	$\frac{3}{8}\gamma_{11}+\frac{1}{8}\gamma_{12}$	$\frac{3}{8}\gamma_{21}+\frac{1}{8}\gamma_{22}$	$\frac{3}{8}\gamma_{31}+\frac{1}{8}\gamma_{32}$

The probability that x is covered, given Case 2, is

$$\begin{aligned}
P(x \text{ covered} \mid \text{Case 1}) &= P_{001}\left(\frac{3}{8}\gamma_{11}+\frac{1}{8}\gamma_{12}\right) \\
&\quad + P_{010}\left(\frac{3}{8}\gamma_{21}+\frac{1}{8}\gamma_{22}\right) + P_{100}\left(\frac{3}{8}\gamma_{31}+\frac{1}{8}\gamma_{32}\right) \\
&\quad + P_{011}\left[1-\left(1-\frac{3}{8}\gamma_{11}-\frac{1}{8}\gamma_{12}\right)\left(1-\frac{3}{8}\gamma_{21}-\frac{1}{8}\gamma_{22}\right)\right] \\
&\quad + P_{110}\left[1-\left(1-\frac{3}{8}\gamma_{21}-\frac{1}{8}\gamma_{22}\right)\left(1-\frac{3}{8}\gamma_{31}-\frac{1}{8}\gamma_{32}\right)\right] \\
&\quad + (P_{111}+P_Q)\left[1-\left(1-\frac{3}{8}\gamma_{11}-\frac{1}{8}\gamma_{12}\right)\left(1-\frac{3}{8}\gamma_{21}-\frac{1}{8}\gamma_{22}\right)\right. \\
&\quad \left.\left(1-\frac{3}{8}\gamma_{31}-\frac{1}{8}\gamma_{32}\right)\right] \\
&= 0.2697
\end{aligned}$$

Now, we are ready to compute the average length of the city that is covered at a random time. Note that there are 3 miles of the city that fit Case 1, namely the three 1-mile strips centered at each home location. At any point on these 3 miles, the probability of coverage is 0.5902. The remaining 3 miles of the city fit Case 2. The probability of coverage for any point along these remaining 3 miles is 0.2697. Let "$\int_{\text{Case k}}$" denote that we integrate over points along the city that fit Case k, for $k \in \{1,2\}$. Therefore, the average length of the city that is covered at a random time is given by

$$\begin{aligned}
E\left[\int_0^6 f(x)dx\right] &= \int_0^6 P(x \text{ covered})dx \\
&= 0.5902\int_{\text{Case 1}} dx + 0.2697\int_{\text{Case 2}} dx \\
&= 3\,(0.5902+0.2697) \;=\; 2.5797
\end{aligned}$$

Problem 5

a) We are in the case of congestion pricing.

Type 1 customers: $\lambda_1 = 30$ /hr and $c_1 = \$2/\text{min} = \$120/\text{hr}$

Type 2 customers: $\lambda_2 = 24$ /hr and $c_2 = \$3/\text{min} = \$180/\text{hr}$

Thus $\lambda = \lambda_1 + \lambda_2 = 54/\text{hr}$.

And for both types of customers: E[S] = 1 min = $\frac{1}{60}$ hr and $\sigma_S = 0$.

The total cost is given by C = $\bar{c}\, W_q$.

With $W_q = \frac{\lambda[E^2[S] + \sigma_S^2]}{2(1-\rho)} = \frac{54(\frac{1}{60^2} + 0)}{2(1-\frac{54}{60})} = 0.075$ hr.

Therefore, the cost is:

C = $(c_1\lambda_1 + c_2\lambda_2)W_q = (120\times 30 + 180\times 24)\times 0.075 = \$594$

b) The marginal cost MC_1 for Type 1 customers is given by

$$MC_1 = \frac{dC}{d\lambda_1} = c_1 W_q + \bar{c}\frac{dW_q}{d\lambda}$$

The first term gives the internal cost, and we have $c_1 W_q = \$9/\text{hr}$.

The second term gives the external cost and we have:

$$\bar{c}\frac{dW_q}{d\lambda} = (c_1\lambda_1 + c_2\lambda_2)[\frac{E^2[S]}{2(1-\rho)} + \frac{1}{\mu}\frac{\lambda E^2[S]}{2(1-\rho)}] = \$110/\text{hr}.$$

For Type 2 customers, we have $MC_2 = \frac{dC}{d\lambda_2} = c_2 W_q + \bar{c}\frac{dW_q}{d\lambda}$.

The internal cost is $c_2 W_q = \$13.5$ /hr.

The external cost $\bar{c}\frac{dW_q}{d\lambda}$ is the same as for Type 1 customers: $\bar{c}\frac{dW_q}{d\lambda} = \$110/\text{hr}$.

The external cost is the same because the service time for the two types of customers is the same.

c) Let S_1 be service time of Type 1 customers: E[S_1] = $\frac{1}{\mu_1} = 0.5$ min = $\frac{1}{120}$ hr.

Let S_2 be service time of Type 2 customers: E[S_2] = $\frac{1}{\mu_2} = 1.625$ min = $\frac{13}{480}$ hr.

For the entire set of facility users, we have:

$\lambda = 54/\text{hr}$ $\qquad\qquad$ $\frac{1}{\mu} = \text{E[S]} = \frac{\lambda_1}{\lambda}\frac{1}{\mu_1} + \frac{\lambda_2}{\lambda}\frac{1}{\mu_2} = \frac{1}{60}$ hr

$$\rho = \frac{\lambda}{\mu} = 0.9 \qquad \text{c} = \frac{\lambda_1}{\lambda} c_1 + \frac{\lambda_2}{\lambda} c_2 = \frac{440}{3} \text{ \$/hr}$$

The total cost is given by $C = c\lambda W_q$, with $W_q = \frac{\lambda[E^2[S] + \sigma_S^2]}{2(1-\rho)}$.

We have $E[S^2] = \frac{\lambda_1}{\lambda}\frac{1}{\mu_1^2} + \frac{\lambda_2}{\lambda}\frac{1}{\mu_2^2} = \frac{7}{19200}$ hr, thus the waiting time is given by:

$$W_q = \frac{\lambda[E^2[S] + \sigma_S^2]}{2(1-\rho)} = \frac{\lambda E[S^2]}{2(1-\rho)} = \frac{54\frac{7}{11520}}{2(1-0.9)} = \frac{63}{640},$$

$$\text{and } C = \frac{440}{3} \times 54 \times \frac{63}{640} \approx \$720$$

d) The same calculations as in part b) give us:

Type of customers	Marginal Cost ($)	
	Internal	External
1	11.81	67.72
2	17.72	240.2

Now, the external costs are different. The external cost is higher for Type 2 customers because their service time is higher.

e) Type 1 customers should be assigned priority because the ratio $f_1 = \frac{c_1}{E[S_1]} = 4$ \$/min is greater that Type 2 customers' ration $f_2 = 1.85$ \$/min.

f) Let's compute the total cost of waiting at the facility per hour, given that Type 1 customers have the priority over Type 2 customers.

$C = c_1\lambda_1 W_{q1} + c_2\lambda_2 W_{q2}$

$$W_0 = \frac{\lambda_1 E[S_1^2] + \lambda_2 E[S_2^2]}{2} = \frac{\lambda_1 E^2[S_1] + \lambda_2 E^2[S_2]}{2} = \frac{63}{6400}$$

$$W_{q1} = \frac{W_0}{1-a_1} = \frac{W_0}{1-\rho_1} = \frac{W_0}{1-\frac{\lambda_1}{\mu_1}} = \frac{21}{1600}$$

$$W_{q2} = \frac{W_0}{(1-a_1)(1-a_2)} = \frac{W_0}{(1-\frac{\lambda_1}{\mu_1})(1-\frac{\lambda_1}{\mu_1}-\frac{\lambda_2}{\mu_2})} = \frac{21}{160}$$

Therefore, $C = \frac{2457}{4} = \$614.25$/hr

This is a 14.7% improvement compared to the cost in c).

g) We have the two rations $f_1 = 4$ \$/min and $f_2 = 1.85$ \$/min. Since $f_1 > f_2$, Type 1 customers should be assigned priority, in order to minimize the expected cost of the total time that the all customers spend in the system.

h) The service times are no longer deterministic. Therefore, the expected waiting times will increase.

$W_0 = \frac{\lambda_1 E[S_1^2] + \lambda_2 E[S_2^2]}{2} = \frac{\lambda_1 (E^2[S_1] + \sigma_{S1}^2) + \lambda_2 (E^2[S_2] + \sigma_{S2}^2)}{2} = \frac{63}{3200}$ (This is twice as much as previously.)

$$W_{q1} = \frac{W_0}{1 - a_1} = \frac{W_0}{1 - \rho_1} = \frac{W_0}{1 - \frac{\lambda_1}{\mu_1}} = \frac{21}{800}$$

$$W_{q2} = \frac{W_0}{(1 - a_1)(1 - a_2)} = \frac{W_0}{(1 - \frac{\lambda_1}{\mu_1})(1 - \frac{\lambda_1}{\mu_1} - \frac{\lambda_2}{\mu_2})} = \frac{21}{80}$$

The waiting times are twice as high as in part f), therefore, the cost is going to be twice as important as in f):

C = $c_1 \lambda_1 W_{q1} + c_2 \lambda_2 W_{q2}$ = \$1228.5/hr