1.060 Engineering Mechanics II Spring 2006

Recitation 9 - Problems

April 27th and 28th

Problem 1

Figure 1 shows the cross-section of a circular gate (called *Tainter* gate) of radius $R = 5\ m$. The gate is placed in a rectangular channel of width $b = 50\ m$ and spans the entire width of the channel. The depth upstream of the gate is $h_1 = 6\ m$. The gate is partially opened, leaving a gap of height $h_g = 1.3\ m$ between the gate and the bottom of the channel. The discharge under the gate per unit width of the channel is $q = 10\ m^2/s$.

a) Determine the depth of flow h_2, a short distance downstream of the gate opening.

b) Determine the horizontal force from the fluid on the gate, per unit width of the gate.

c) Determine the contraction coefficient for the flow under the gate.

d) Classify the flow upstream and downstream of the gate as super- or subcritical.

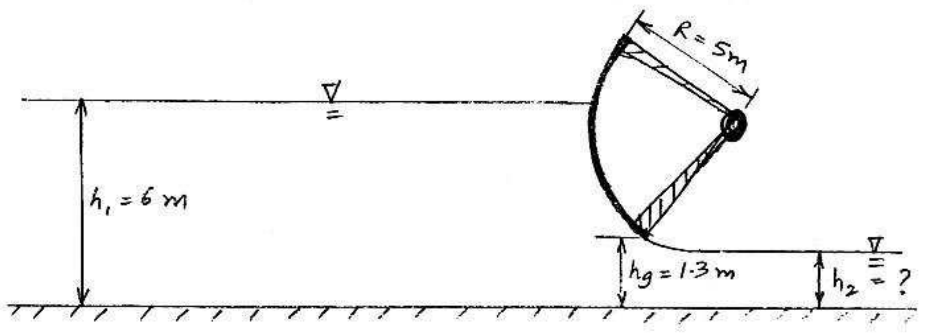


Figure 1: Flow under a Tainter gate in Problem 1.

Problem 2

Figure 2 shows a triangular channel cross-section. For a channel slope $S_0 = 0.01$, a discharge $Q = 50\ m^3/s$, and a Manning's $n = 0.02$ [SI units]:

a) Determine the normal depth, $h = h_n$, corresponding to uniform steady flow, and the associated average velocity, $V = V_n$.

b) Show that normal flow is supercritical.

c) Determine the critical depth, $h = h_c$.

Due to the presence of a gate downstream, a hydraulic jump takes place in the channel, in which the depth transitions from $h_1 = h_n$ to a new value of the depth, h_2.

(Please turn over.)

d) Determine the value of the depth h_2 and the corresponding average velocity, V_2.

e) Calculate the headloss and the rate of energy dissipation associated with the hydraulic jump.

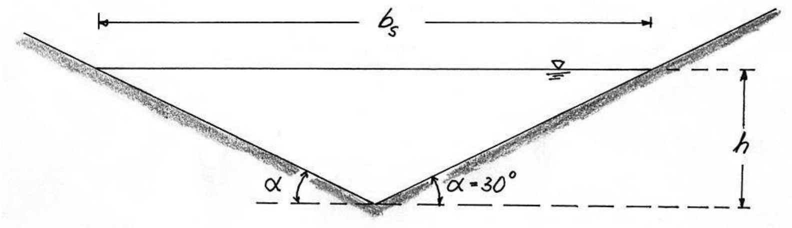


Figure 2: Triangular channel cross-section in Problem 2.

Problem 3

A rectangular channel of width $b = 50\ m$ carries a discharge of $Q = 300\ m^3/s$ and has a Manning's $n = 0.02$ [SI units].

a) If normal depth of flow in the channel is $h_n = 3.0\ m$, determine the slope of the channel.

b) Is the normal flow sub- or supercritical?

The channel is crossed by a bridge, which is supported by two bridge piers, each $b/5 = 10\ m$ wide, and placed in the channel as shown in Figure 3. The obstruction to the flow created by the bridge piers results in a depth of h_1=3.2 m, a relatively short distance upstream of the piers (at 1-1), whereas the depth downstream of the piers (at 2-2) is $h_2 = 3.0\ m$ (i.e., equal to the normal depth). The bottom slope may be assumed sufficiently small to neglect differences in bottom elevation over the short distance between 1-1 and 2-2 and, consistent with this assumption, we neglect shear stresses acting on the wetted perimeter of the channel between 1-1 and 2-2.

c) Determine the force, F_P, exerted by the flow on each of the two bridge piers.

d) Determine the headloss, ΔH_P, from section 1-1 to 2-2.

e) Establish an equation for the depth h_M at the point denoted by M in the sketch and determine h_M.

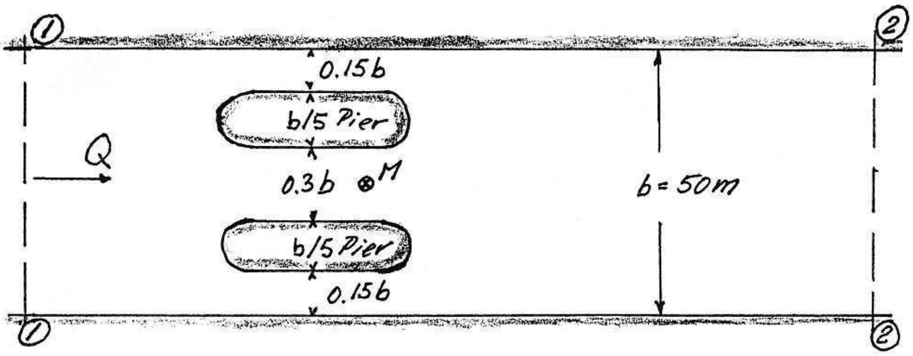
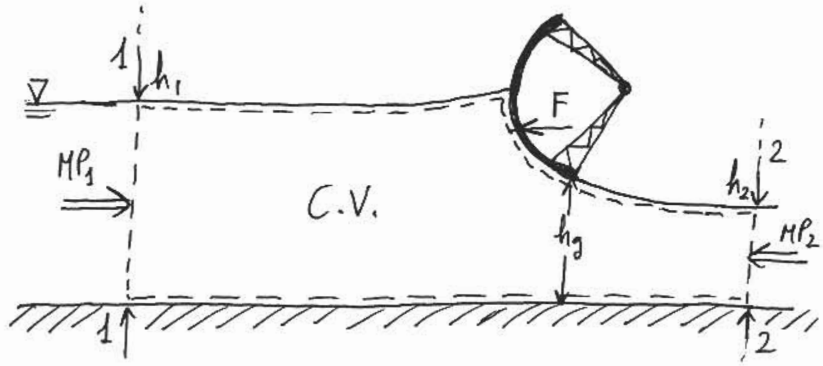


Figure 3: Bridge piers in a rectangular channel in Problem 3.

RECITATION 9 - SOLUTIONS

- PROBLEM N° 1 :



a) Short transition of converging flow $\Rightarrow$ Energy is conserved $\Rightarrow H_1 = H_2 \Rightarrow$

$$\Rightarrow \frac{V_1^2}{2g} + \underbrace{\frac{p_1}{\rho g} + z_1}_{= h_1 + \cancel{z_{BOTTOM}}} = \frac{V_2^2}{2g} + \underbrace{\frac{p_2}{\rho g} + z_2}_{= h_2 + \cancel{z_{BOTTOM}}} \quad (1)$$

$\rightarrow$ B/C flow is well-behaved in 1-1 & 2-2.

Continuity $\Rightarrow q = V_1 h_1 = V_2 h_2 \Rightarrow V_1 = q/h_1$, $V_2 = q/h_2$ (2)

Plug (2) into (1) : $\frac{q^2}{2g}\frac{1}{h_1^2} + h_1 = \frac{q^2}{2g}\frac{1}{h_2^2} + h_2 \Rightarrow 6'142 = \frac{5'102}{h_2^2} + h_2$ (3)

Equation (3) has three solutions : A negative one (with no physical meaning), $h_2 = h_1$ (trivial), and h_2 = alternate depth of h_1. We are interested on the latter. Since we expect $h_2 < h_1$, we are seeking for the supercritical solution. Therefore, the kinetic term, $\frac{q^2}{2g}\frac{1}{h_2^2} = \frac{5'102}{h_2^2}$ (s.i.) is more important than the elevation term h_2, and has to be "singled out" to iterate, i.e.,

$$\frac{5'102}{h_2^2} = 6'142 - h_2 \Rightarrow h_{2,k+1} = \sqrt{\frac{5'102}{6'142 - h_{2,k}}}$$

Take as initial guess, e.g., $h_{2,0} = 0$. Then:

k	0	1	2	3	4
$h_{2,k}$	0'911	0'988	0'995	0'996	0'996

$\Rightarrow \underline{h_2 \approx 1'00 \text{ m}}$

ALTERNATIVE METHOD, WHICH ALMOST ALWAYS WORKS $\rightarrow$ NEWTON'S METHOD

(3) $\Rightarrow \quad f(h_2) = h_2^3 - 6'142 h_2^2 + 5'102 = 0$

Solve using $h_{2,k+1} = h_{2,k} - \frac{f(h_{2,k})}{f'(h_{2,k})} = h_{2,k} - \frac{(h_{2,k}^3 - 6'142 h_{2,k}^2 + 5'102)}{(3h_{2,k}^2 - 12'284 h_{2,k})}$

b) Consider the C.V. shown. Balance of horizontal forces yields:

$$MP_1 = MP_2 + F \Rightarrow F = MP_1 - MP_2$$

$$MP_1 = (\rho V_1^2 + p_{CG,1})A_1 = \left(\rho \frac{q^2}{h_1^2} + \rho g \frac{h_1}{2}\right)(b h_1) = 9'65 \cdot 10^6 \text{ N}$$

$$MP_2 = (\rho V_2^2 + p_{CG,2})A_2 = \left(\rho \frac{q^2}{h_2^2} + \rho g \frac{h_2}{2}\right)(b h_2) = 5'25 \cdot 10^6 \text{ N}$$

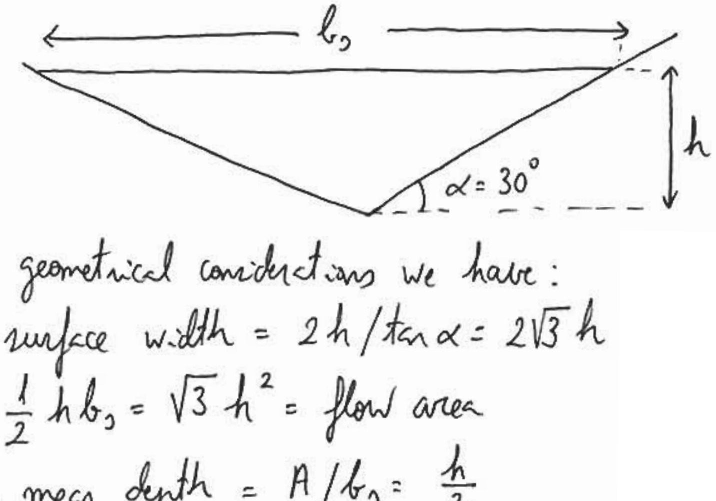
$\underline{\underline{F}} = MP_1 - MP_2 = \underline{\underline{4'4 \cdot 10^6 \text{ N}}}$ (The force on the C.V. acts towards the left; the force on the gate acts towards the right).

c) $\underline{\underline{C_c}} = \frac{h_2}{h_g} = \frac{1}{1'3} = \underline{\underline{0'77}}$ ($> 0'61$, which makes sense, because the Tainter gate has a smoother shape than a sharp vertical gate).

d) Upstream: $Fr_1 = \frac{V_1}{\sqrt{g h_1}} = 0'22 < 1 \rightarrow$ Subcritical flow.

Downstream: $Fr_2 = \frac{V_2}{\sqrt{g h_2}} = 3'19 > 1 \rightarrow$ Supercritical flow.

-PROBLEM N° 2:



From geometrical considerations we have:

b_s = surface width $= 2h/\tan\alpha = 2\sqrt{3}\,h$

$A = \frac{1}{2} h b_s = \sqrt{3}\,h^2$ = flow area

h_m = mean depth $= A/b_s = \frac{h}{2}$

P = wetted perimeter $= 2h/\sin\alpha = 4h$

R_h = hydraulic radius $= A/P = (\sqrt{3}/4)\,h$

a) Manning's equation gives

$$Q = VA = \frac{1}{n} R_h^{2/3} \sqrt{S_0}\, A = \frac{1}{n}\left(\frac{\sqrt{3}}{4} h_n\right)^{2/3} S_0^{1/2}\, \sqrt{3} h_n^2 \Rightarrow h_n^{8/3} = (nQ/S_0^{1/2})\left(\frac{\sqrt{3}}{4}\right)^{-2/3} \frac{1}{\sqrt{3}} \Rightarrow$$

$$\Rightarrow h_n^{8/3} = (0'02 \cdot 50/\sqrt{0'01})\left(\frac{\sqrt{3}}{4}\right)^{-2/3}\frac{1}{\sqrt{3}} \Rightarrow h_n^{8/3} = 10'09 \Rightarrow \underline{\underline{h_n = 2'38\,m}}$$

b)

$$V = Q/A = 50/(\sqrt{3}\cdot 2'38^2) = 5'10\ m/s\ ;\ h_m = \frac{1}{2} h_n = 1'19\,m\ ;\ Fr = \frac{V}{\sqrt{g h_m}} = 1'49 > 1 \Rightarrow$$

$\Rightarrow$ SUPERCRITICAL FLOW

c) Corresponding to critical flow we have $V = V_c = \sqrt{g h_{mc}} = \sqrt{\frac{1}{2} g h_c}$;

$A = A_c = \sqrt{3} h_c^2$. So,

$$Q = V_c A_c = \sqrt{\tfrac{1}{2} g h_c}\, \sqrt{3} h_c^2 = \sqrt{\tfrac{3}{2} g}\ h_c^{5/2} \Rightarrow \underline{\underline{h_c}} = \left(\frac{Q}{\sqrt{\frac{3}{2} g}}\right)^{2/5} = \left(\frac{50}{\sqrt{\frac{3}{2} 9'8}}\right)^{2/5} = \underline{\underline{2'79\,m}}$$

(Note: $h_c > h_n$ in (a): Further evidence that normal flow is supercritical).

d) Hydraulic jumps are analyzed based on the Momentum Principle. We have,

$$MP = \rho V^2 A + p_{CG} A = \rho \frac{Q^2}{A} + \rho g (h - y_{CG}) A$$

For a triangular channel cross-section we have $A = \sqrt{3} h^2$ and $y_{CG} = \frac{2}{3} h$, so

$$MP = \rho \frac{Q^2}{\sqrt{3} h^2} + \rho g \left(h - \frac{2}{3} h\right)\sqrt{3} h^2 = \frac{\rho g}{\sqrt{3}}\left(\frac{Q^2}{g h^2} + h^3\right)$$

The jump condition therefore is $MP_1 = MP_2$, or

$$\frac{Q^2}{g h_1^2} + h_1^3 = \frac{Q^2}{g h_2^2} + h_2^3 \Rightarrow h_2^3 + \frac{255}{h_2^2} = h_n^3 + \frac{255}{h_n^2} = 58'5\ (m^3)$$

Hydraulic jumps only occur from supercritical to subcritical flow. Hence $h_2 > h_c$ and the hydrostatic pressure term is the most important term in MP. Therefore, we arrange the previous equation as ⊛

$$h_{2,k+1}^3 = 58'5 - \frac{255}{h_{2,k}^2}\ ;$$ Start to iterate with $h_{2,0} = h_c + ? = 3\,m$ (for instance),

to obtain: $3 \to 3'11 \to 3'22 \to 3'24 \to 3'24 \to 3'25 \to 3'25 \to 3'25 \Rightarrow \underline{\underline{h_2 = 3'25\,m}}$

(Check: $Fr^2 = (Q^2/A^2)/(g h_m) = (2'73)^2/15'9 = 0'47 < 1$ ✓)

e)

$$H_1 = z_{BOTTOM} + h_1 + \frac{V_1^2}{2g} = z_{BOT.} + h_n + \frac{(Q/A_n)^2}{2g} = z_{BOT.} + 3'705\,(m)$$

$$H_2 = z_{BOTTOM} + h_2 + \frac{V_2^2}{2g} = z_{BOT.} + 3'631\,(m)$$

$\underline{\underline{\Delta H}} = H_1 - H_2 = \underline{0'074\ m}$

$\underline{\underline{\dot{E}}} = \rho g Q \Delta H = 36\ kJ/s = \underline{36\ kW}$

⊛: Alternatively, use Newton's method:

$$h_{2,k+1} = h_{2,k} - \frac{(h_{2,k}^5 - 58'5 h_{2,k}^2 + 255)}{(5 h_{2,k}^4 - 117 h_{2,k})}$$

(It's faster!)

– PROBLEM Nº 3:

a) $$V_n = \frac{Q}{b h_n} = \frac{1}{n} R_h^{2/3} S_o^{1/2} = \frac{1}{n}\left(\frac{b h_n}{b+2h_n}\right)^{2/3} S_o^{1/2} \Rightarrow \underline{\underline{S_o}} = \left(\frac{n Q (b+2h_n)^{2/3}}{(b h_n)^{5/3}}\right)^2 = \underline{\underline{4'3 \cdot 10^{-4}}}$$

b) $$V_n = \frac{Q}{b h_n} = \frac{300}{50 \cdot 3} = 2'0 \text{ m/s} \; ; \; Fr_n \underset{\substack{\uparrow \\ \text{RECTANGULAR} \\ \text{CHANNEL}}}{=} \frac{V_n}{\sqrt{g h_n}} = \frac{2'0}{\sqrt{9'8 \cdot 3}} = 0'37 < 1 \Rightarrow$$ NORMAL FLOW IS SUBCRITICAL

c) Momentum balance for fluid between 1-1 & 2-2:

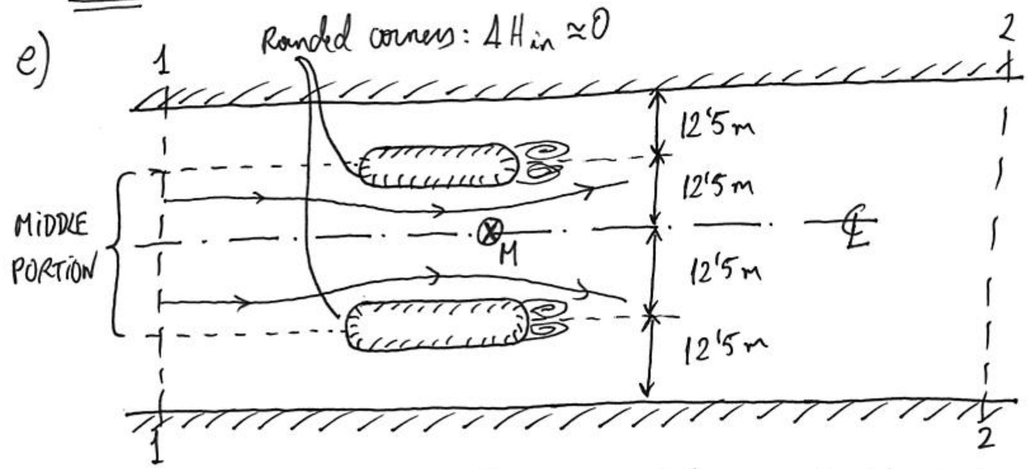
$MP_1 = MP_2 + 2F_P \Rightarrow F_P = \frac{1}{2}(MP_1 - MP_2)$ ($2F_P$ since we have two piers)

$$\underline{\underline{F_P}} = \frac{1}{2}\left[\left(\rho V_1^2 + \frac{1}{2}\rho g h_1\right) h_1 b - \left(\rho V_2^2 + \frac{1}{2}\rho g h_2\right) h_2 b\right] = \underline{\underline{1'35 \cdot 10^5 \text{ N}}}$$

(Obtained with $V_1 = 1'88$ m/s, $h_1 = 3'2$ m, $V_2 = 2'0$ m/s, $h_2 = 3'0$ m)

d) $$\underline{\underline{\Delta H_P}} = H_1 - H_2 = V_1^2/(2g) + h_1 - \left(V_2^2/(2g) + h_2\right) = 3'38 - 3'20 = \underline{\underline{0'18 \text{ m}}}$$

e)



Since the bridge piers have "rounded corners", the "entrance loss" for the flow entering the gaps between the piers is negligible. Thus, the flow from 1-1 to point M is a short transition of a converging flow ⇒

$\Rightarrow \Delta H_{1 \text{ to } M} = 0$, or $\quad \dfrac{V_1^2}{2g} + h_1 = \dfrac{V_M^2}{2g} + h_M$

Due to symmetry of pier location (centerline to either sidewall have a pier in the middle), the flow entering the middle portion of 1-1 (of width $b/2 = 0'5b$) must pass between the piers (width $0'3b$), so from continuity

$$V_1 h_1 \frac{b}{2} = V_M h_M \, 0'3 b \Rightarrow V_M = \frac{5}{3} V_1 \frac{h_1}{h_M}$$

$$\frac{V_1^2}{2g} + h_1 = 3'38 = \left(\frac{5}{3}\right)^2 \frac{V_1^2}{2g}\left(\frac{h_1}{h_M}\right)^2 + h_M = 0'5\left(\frac{h_1}{h_M}\right)^2 + h_M$$

Subcritical flow: Single out elevation term to iterate (or use Newton's method):

$$h_{M,k+1} = 3'38 - 0'5\,(h_1/h_M)^2 = 3'38 - 5'12/h_M^2 \quad (h_M \text{ in m})$$

Start iteration with $h_M \leq h_1$, say, $h_M = h_1 = 3'0$ m, to get $\underline{\underline{h_M = 2'65 \text{ m}}}$.