LECTURE #9

1.060 ENGINEERING MECHANICS II

REYNOLDS TRANSPORT THEOREM

We have derived volume conservation in terms of bulk flow description $Q = VA$ = constant; but if we need details we need to do a lot of work, e.g. draw flow nets to get details on V(elocity) and then use Bernoulli to get details on p(ressure). Then we have to integrate p over a surface to get total pressure force. [and we don't get any information about shear forces!]. We want to get other fluid quantities in terms of their BULK values, like Q, but not the details. To do this we take: FINITE CONTROL VOLUME

Let "m" be a fluid property per unit volume of fluid. With finite volume $\forall$ we then have a total amount of "m"

$$M = \int_{\forall} m \, d\forall$$

The rate of change of M for this volume (consisting of the same molecules) is

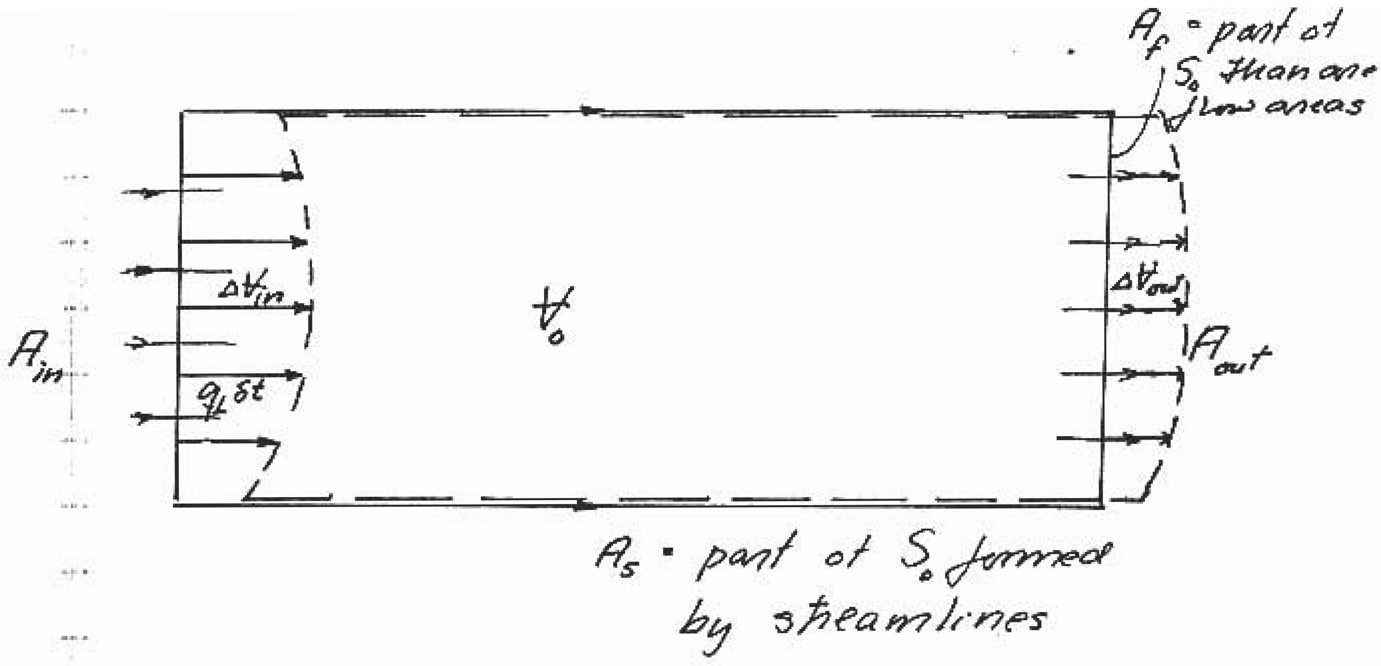
$$\frac{DM}{Dt} = \lim_{\delta t \to 0} \frac{M(t_o + \delta t) - M(t_o)}{\delta t} = \text{Total or Material Derivative}$$

where

$$M(t_o) = \int_{\forall(t_o)} m(t_o) \, d\forall$$

and

$$M(t_o + \delta t) = \int_{\forall(t_o + \delta t)} m(t_o + \delta t) \, d\forall = \int_{\forall(t_o + \delta t)} \left(m(t_o) + \frac{\partial m}{\partial t} \delta t + \right) d\forall$$



$$\forall_o = \forall(t_o) \; ; \quad M_o = M(t_o) = \int_{\forall_o} m_o \, d\forall$$

$$\forall(t_o + \delta t) = \forall_o - \Delta\forall_{in} + \Delta\forall_{out}$$

$$\Delta\forall_{in} = \int_{A_{in}} (q_\perp \delta t)\, dA = \left(\int_{A_{in}} q_\perp \, dA\right)\delta t \; ; \; \Delta\forall_{out} = \left(\int_{A_{out}} q_\perp \, dA\right)\delta t$$

$$M(t_o + \delta t) = \int_{\forall_o} m_o \, d\forall + \left(\int_{\forall_o} \frac{\partial m}{\partial t} d\forall\right)\delta t + (\delta t)^2 \text{ from } \Delta\forall\text{'s}$$

$$- \left(\int_{A_{in}} m_o q_\perp \, dA\right)\delta t + \left(\int_{A_{out}} m_o q_\perp \, dA\right)\delta t + \delta t^2 \text{ from } \frac{\partial m}{\partial t}$$

$$M(t_o + \delta t) = (\text{"m" in } \forall_o \text{ @ } t_o + \delta t) - (\text{"m" in } \Delta\forall_{in}) + (\text{"m" in } \Delta\forall_{out})$$

$$\frac{DM}{Dt} = \int_{\forall_o} \frac{\partial m}{\partial t} d\forall - \int_{A_{in}} m q_\perp \, dA + \int_{A_{out}} m q_\perp \, dA =$$

$$\frac{\partial}{\partial t}\int_{\forall_o} m \, d\forall - \int_{A_{in}} m q_\perp \, dA + \int_{A_{out}} m q_\perp \, dA$$

IN WORDS

Rate of change of M for a volume following the fluid (same fluid particles within volume at all times) =

Rate of change of M within volume between fixed in- and outflow areas [The CONTROL VOLUME]

− Rate of inflow of M into CONTROL VOLUME

+ Rate of outflow of M from CONTROL VOLUME

Try this out for our old friend mass conservation.

Since ~~M~~ = ~~ρ~~ mass/volume, and M as we move with the fluid is constant, we have

$$\frac{DM}{Dt} = 0 = \underbrace{\frac{\partial}{\partial t}\int_{V_0} \rho \, dV}_{\partial M/\partial t} - \underbrace{\int_{A_{in}} \rho q_\perp \, dA}_{\dot{M}_{in}} + \underbrace{\int_{A_{out}} \rho q_\perp \, dA}_{\dot{M}_{out}}$$

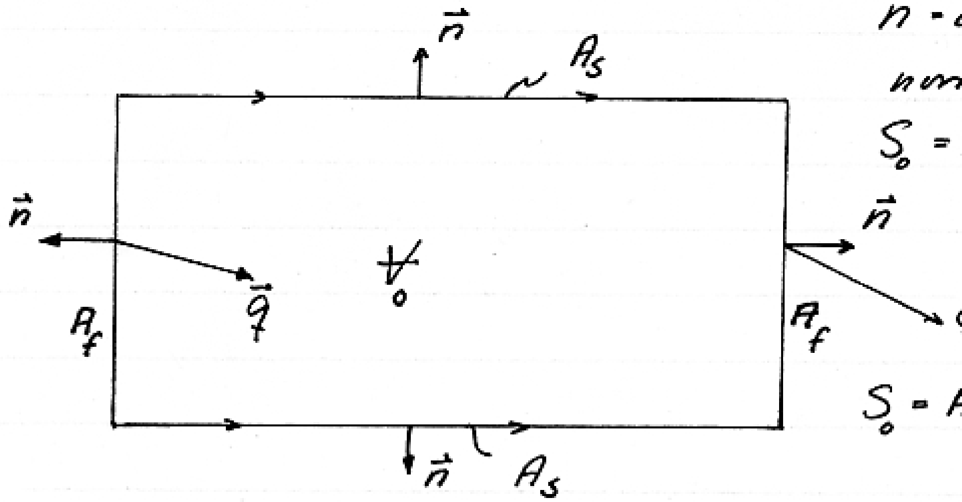
$$\underline{\dot{M}_{in} - \dot{M}_{out} = \frac{\partial M}{\partial t}}$$ "old hat" (Lecture #5)

For volume itself − "m" = unity. If fluid is incompressible volume is conserved, and

$$\frac{DV}{Dt} = 0 = \underbrace{\frac{\partial}{\partial t}(V)}_{} - \underbrace{\int_{A_{in}} q_\perp \, dA}_{Q_{in}} + \underbrace{\int_{A_{out}} q_\perp \, dA}_{Q_{out}}$$

$$Q_{in} - Q_{out} = \frac{\partial V}{\partial t}$$ "even older hat" (Lecture #5)

We can compact this expression, known as the Reynolds Transport Theorem by the following "trick"



$\vec{n}$ - unit outward normal to S_o

S_o = surface enclosing $\forall_o$

$S_o = A_s + A_f$

At inflow: $q_\perp = -\vec{n}\vec{q}$, $\vec{n}\cdot\vec{q} = -q_\perp$ @ A_{in}

Along streamline, A_s: $q_\perp = \vec{n}\vec{q} = 0$, $q_\perp = 0$ @ A_s

At outflow: $q_\perp = \vec{n}\cdot\vec{q}$, $\vec{n}\cdot\vec{q} = q_\perp$ @ A_{out}

$$\frac{DM}{Dt} = \frac{\partial}{\partial t}\int_{\forall_o} m\, d\forall + \int_{S_o} m\vec{n}\cdot\vec{q}\, dS$$

Here's where we really need it: Conservation of (LINEAR) MOMENTUM, or NEWTONS LAW.

Rate of change of Momentum [for a volume consisting of the same particles] = Sum of Forces on this volume

Linear Momentum per unit volume = $\rho\vec{q} = \vec{m}$

Reynolds Transport Theorem

Rate of change of momentum = $\frac{D\vec{M}}{Dt}$ =

$$\frac{\partial}{\partial t}\int_{\forall_o} \rho\vec{q}\, d\forall + \int_S \rho\vec{q}(\vec{n}\cdot\vec{q})\, dS = \sum\left(\overrightarrow{\text{Forces on } \forall_o}\right)$$