

**1.033/1.57**

**Mechanics of Material Systems**  
(Mechanics and Durability of Solids I)

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*Lecture: MWF1 // Recitation: F 3:00-4:30*

# Part III: Elasticity and Elasticity Bounds

## 6. The Theorem of Virtual Work and Variational Methods in Elasticity

# Content 1.033/1.57

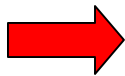
## Part I. **Deformation and Strain**

- 1 Description of Finite Deformation
- 2 Infinitesimal Deformation

## Part II. **Momentum Balance and Stresses**

- 3 Momentum Balance
- 4 Stress States / Failure Criterion

## Part III. **Elasticity and Elasticity Bounds**

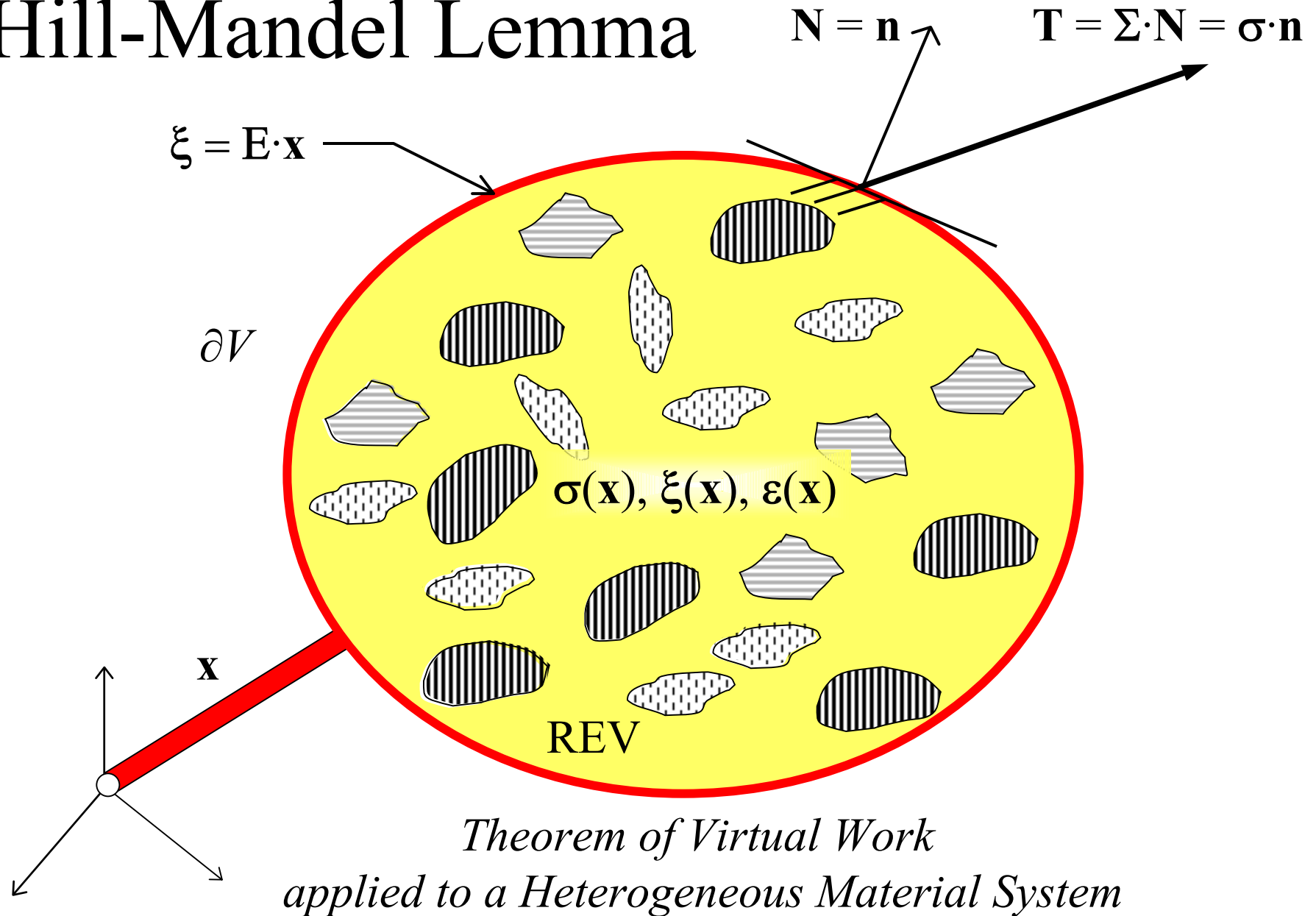


- 5 Thermoelasticity,
- 6 Variational Methods

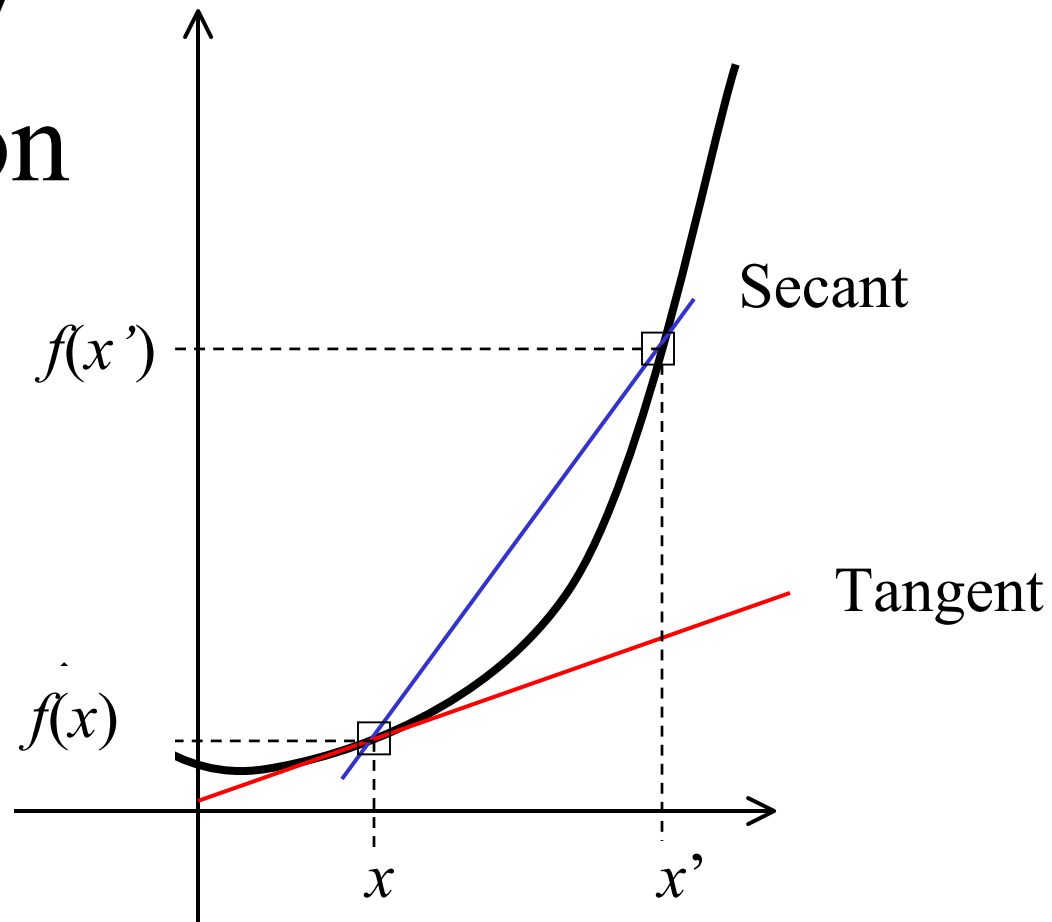
## Part IV. **Plasticity and Yield Design**

- 7 1D-Plasticity – An Energy Approach
- 8 Plasticity Models
- 9 Limit Analysis and Yield Design

# Hill-Mandel Lemma

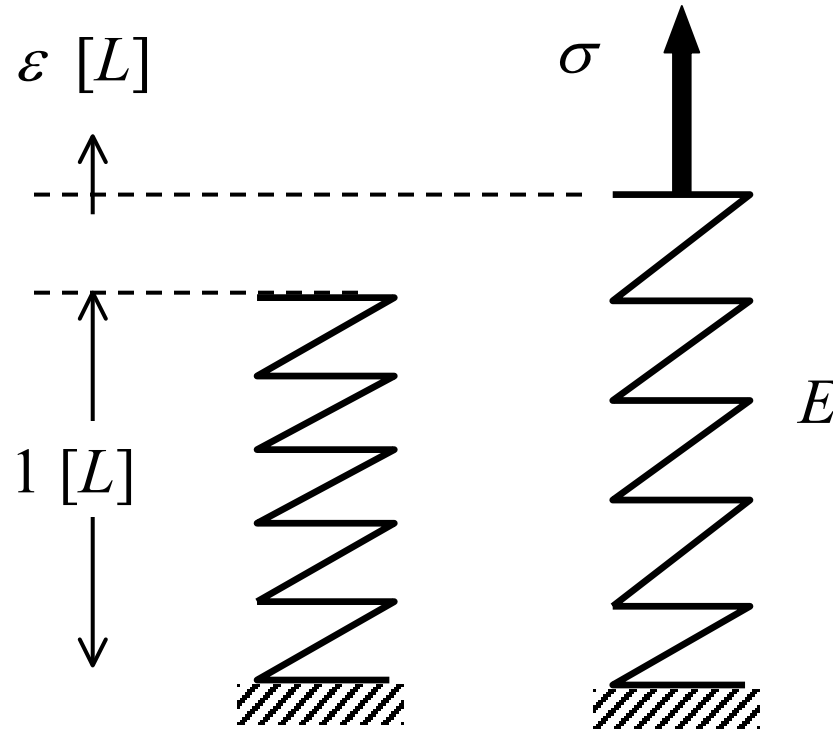


# Convexity of a function



$$\left. \frac{\partial f}{\partial x} \right|_x (x' - x) \leq f(x') - f(x)$$

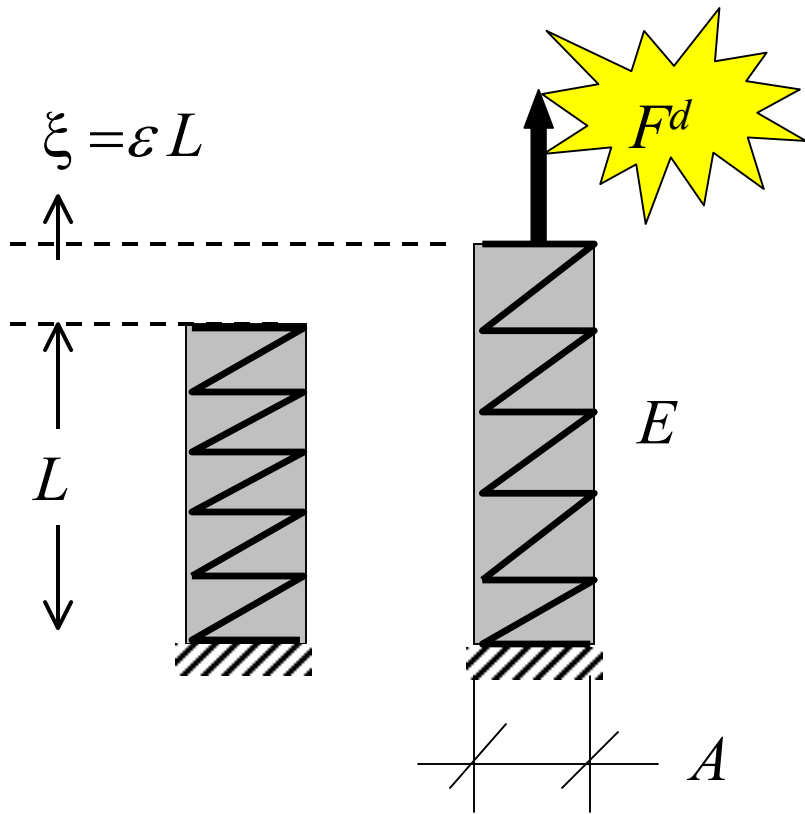
# Convexity: Applied to Free Energy



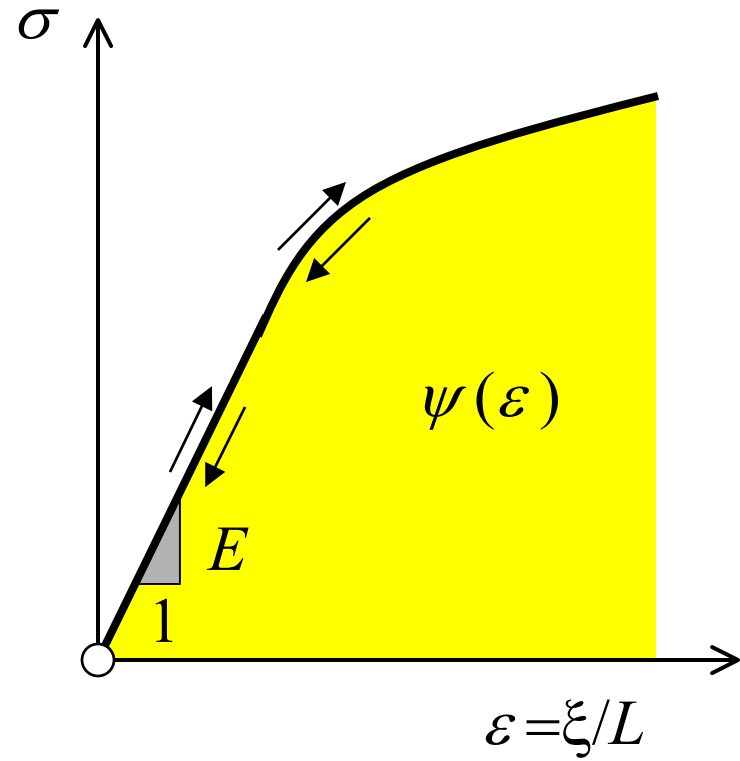
$\sigma$   
State  
Equation

$$\left. \frac{\partial \psi}{\partial \varepsilon} \right|_{\varepsilon} (\varepsilon' - \varepsilon) \leq \psi(\varepsilon') - \psi(\varepsilon)$$

# Theorem of Minimum Potential Energy



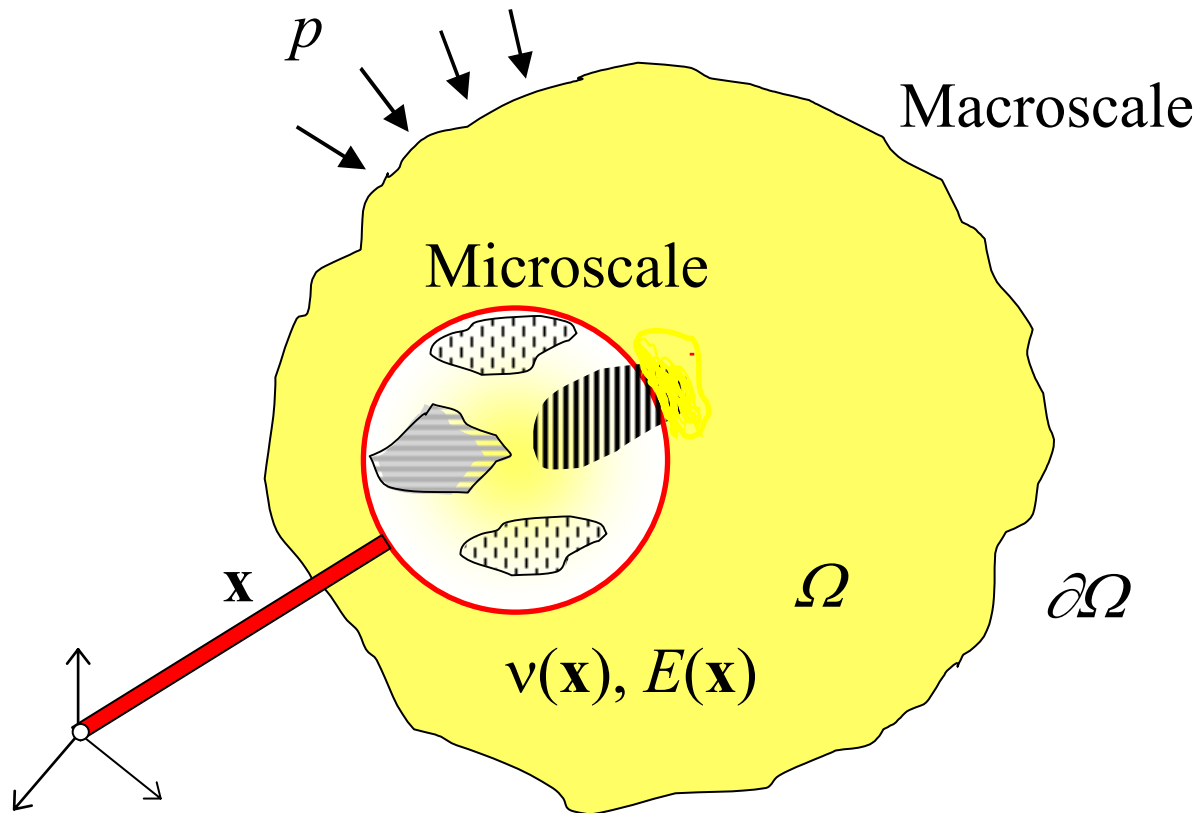
*1-Parameter System*



$$E_{pot}(\xi) = \min_{\xi' \in KA} [W(\epsilon') - \Phi(\xi')]$$

Upper  
Energy  
Bound

# Ex: Heterogeneous Material System I



1. Displacement Field (KA)

$$\xi' = a\mathbf{x}$$

2. Stored Energy

$$W(\varepsilon' = a\mathbf{1})$$

3. External Work

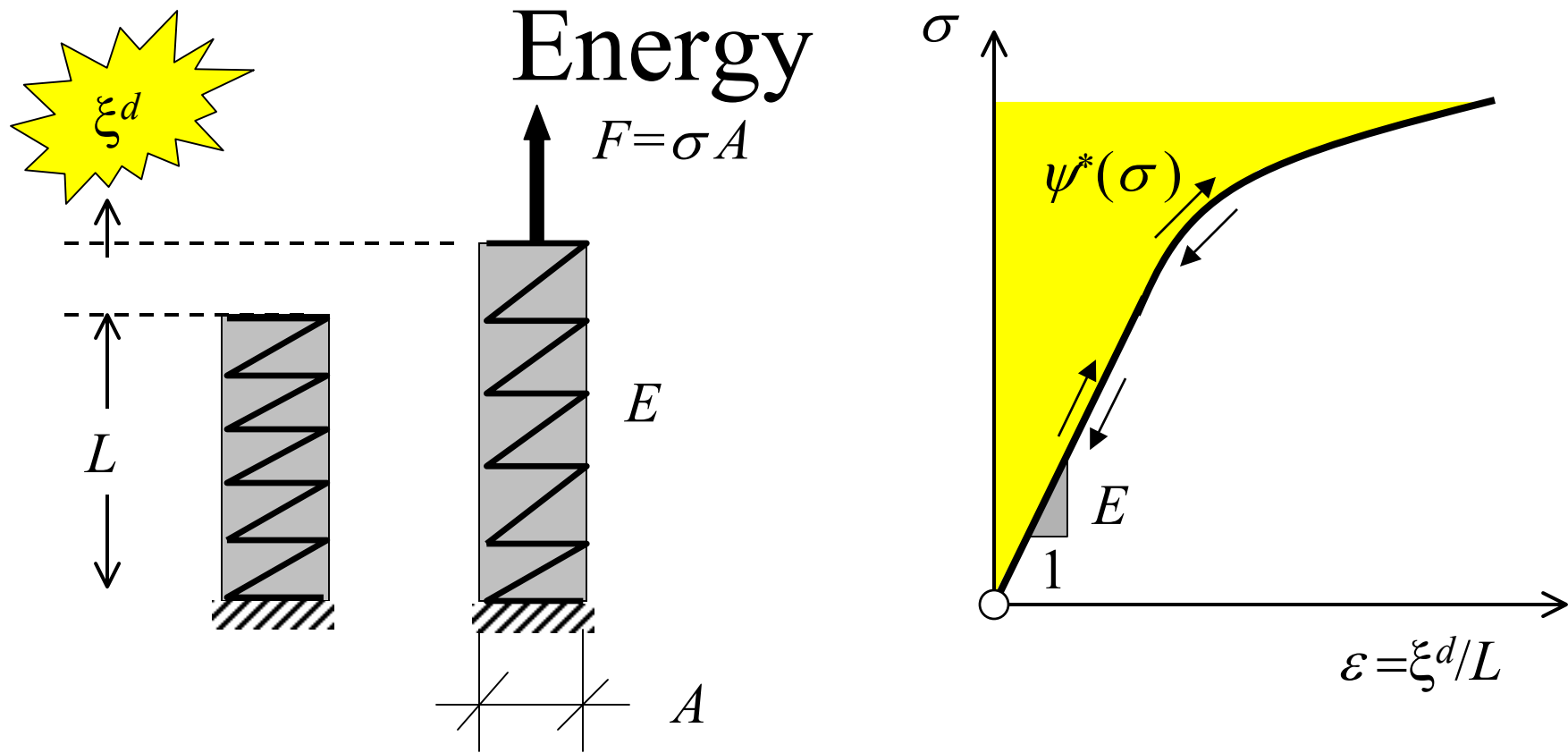
$$\Phi(\xi') = f(T^d)$$

$$E_{pot}(\xi') = W(\varepsilon') - \Phi(\xi')$$

Upper  
Energy  
Bound



# Theorem of Minimum Complementary

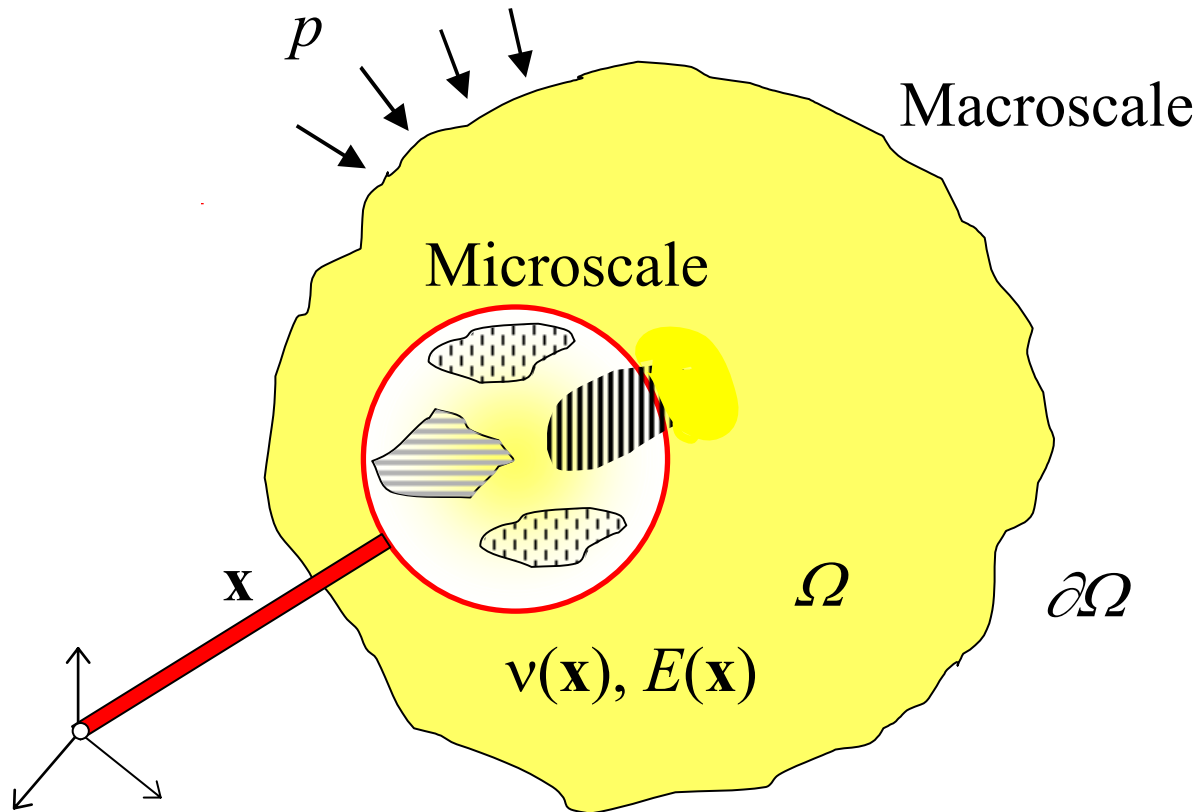


*1-Parameter System*

$$E_{com}(\sigma) = \min_{\sigma' SA} [W^*(\sigma') - \Phi^*(\sigma')]$$

Lower  
Energy  
Bound

# Ex: Heterogeneous Material System II



1. Stress Field (SA)

$$\boldsymbol{\sigma}' = p\mathbf{1}$$

2. Complementary Energy

$$W^*(\boldsymbol{\sigma}' = p\mathbf{1})$$

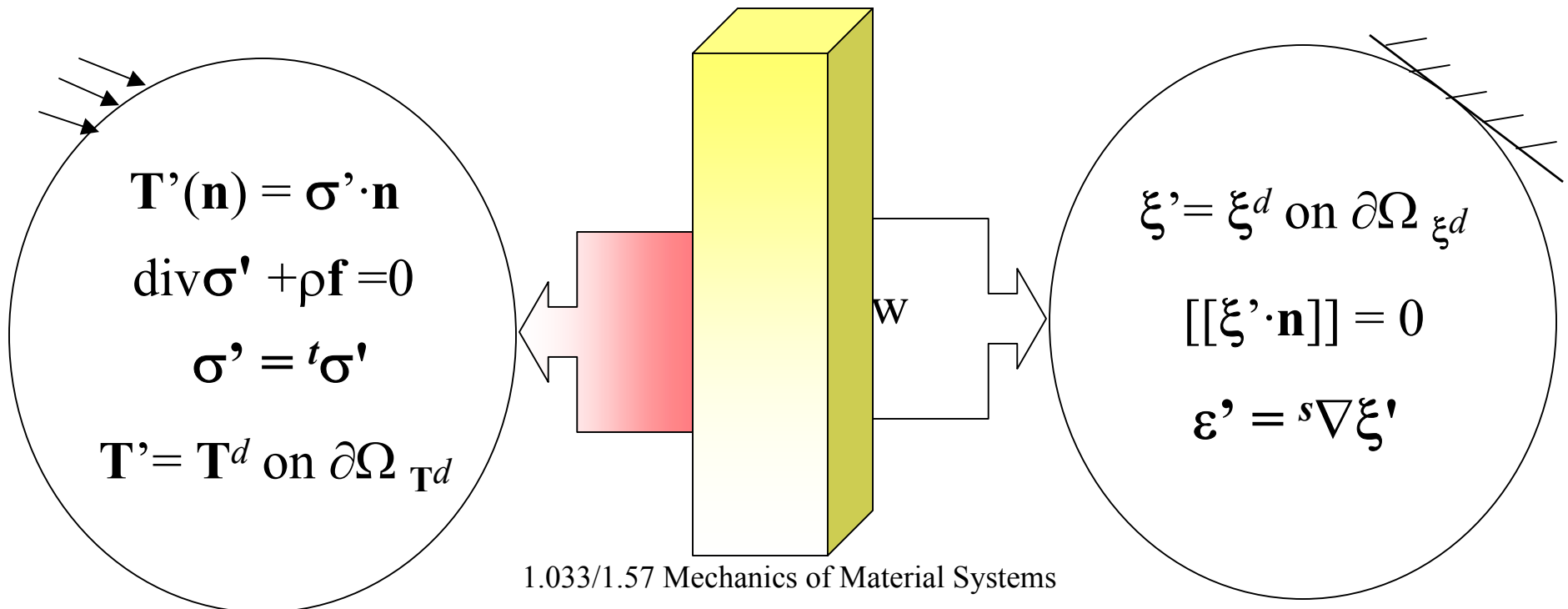
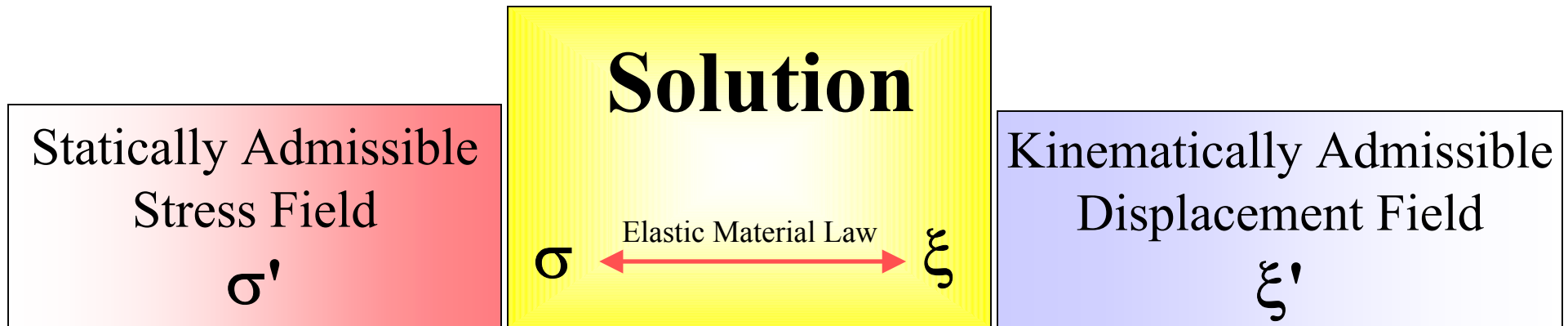
3. External Work

$$\Phi^*(\boldsymbol{\sigma}') = f(\xi^d)$$

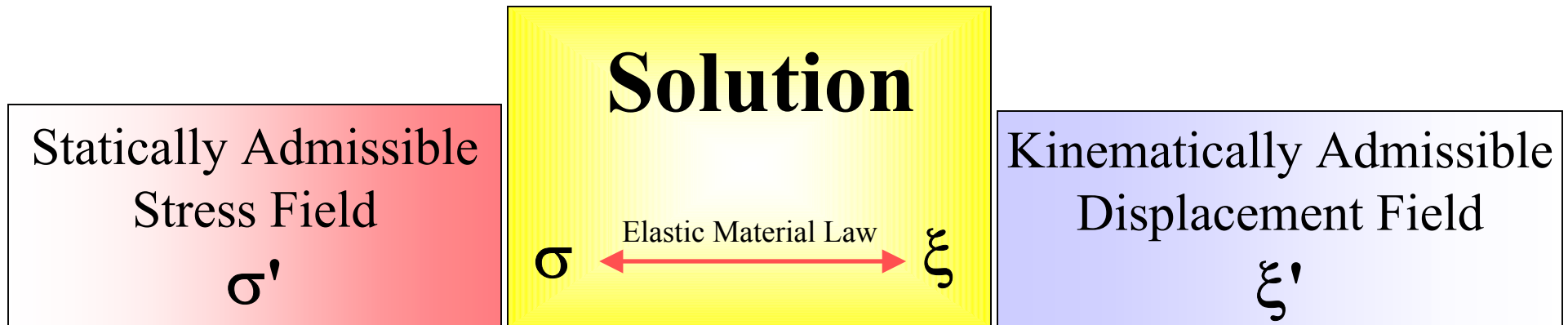
$$E_{com}(\boldsymbol{\sigma}') = W^*(\boldsymbol{\sigma}') - \Phi^*(\boldsymbol{\sigma}')$$

Lower  
Energy  
Bound

# Elements of Elastic Energy Bounds



# Elastic Energy Bounds (Cont'd)



$$\leq -E_{com}(\sigma) = E_{pot}(\xi) \leq$$

$$-E_{com}(\sigma') = -W^*(\sigma') + \Phi^*(\sigma')$$

$$\uparrow$$

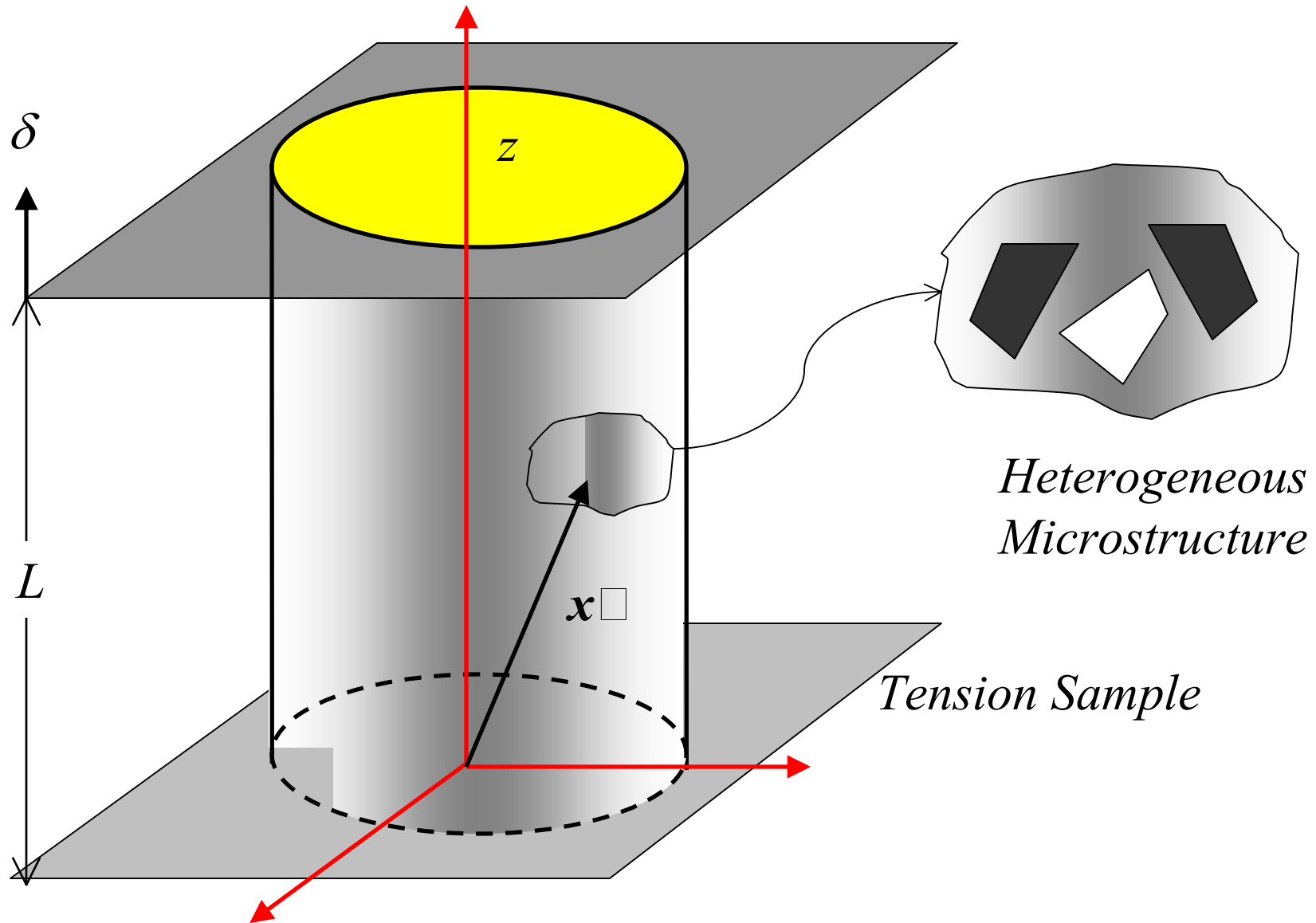
$$\partial\Omega_{\xi d}$$

$$E_{pot}(\xi') = W(\varepsilon') - \Phi(\xi')$$

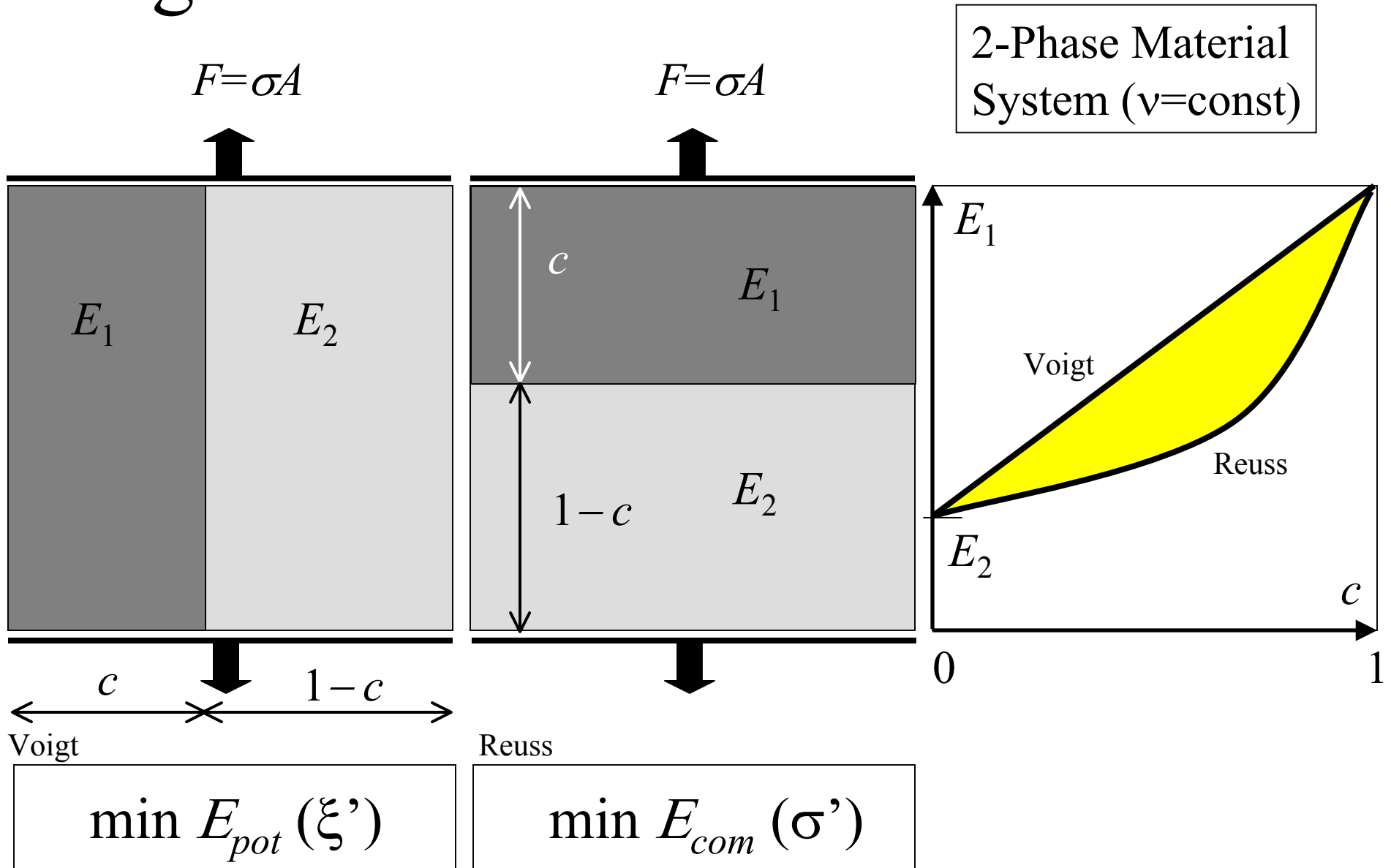
$$\uparrow$$

$$\partial\Omega_{T d}$$

# Training Set: Effective Modulus



# Voigt-Reuss Bounds



# Problem Set Recitation

