1.033/1.57 Q#1: Stress & Strength − Conical Indentation Tests

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Instructor: Franz-Josef ULM

Instrumented nano-indentation is a new technique in materials science and engineering to determine material strengths at very fine scales. We have already studied the 'flat' indentation problem in the homework set. Focus of this exercise is a more realistic shape of intenders, that is the conical indentation test, as sketched in the Figure below. The indenter (which is made of diament) is considered a rigid cone of half-apex angle α. The depth of the indenter in the material is h. A vertical force F is exerted on the rigid indenter in the direction of the Oz-axis. The test gives access to the hardness H of the indented material, defined as the average pressure:

$$H = \frac{F}{A}$$

where $A = \pi R^2$ is the projected contact area at the surface $z = 0$ (see figure). The aim of this exercise is to relate the hardness to the strength properties of the indented material. Given the flat surface of the indenter, the contact of the rigid indenter with the material (along the cone surface oriented by unit normal **n**) can be considered to be without friction. Throughout this exercise we will assume quasi-static conditions (inertia effects neglected), and we will neglect body forces.
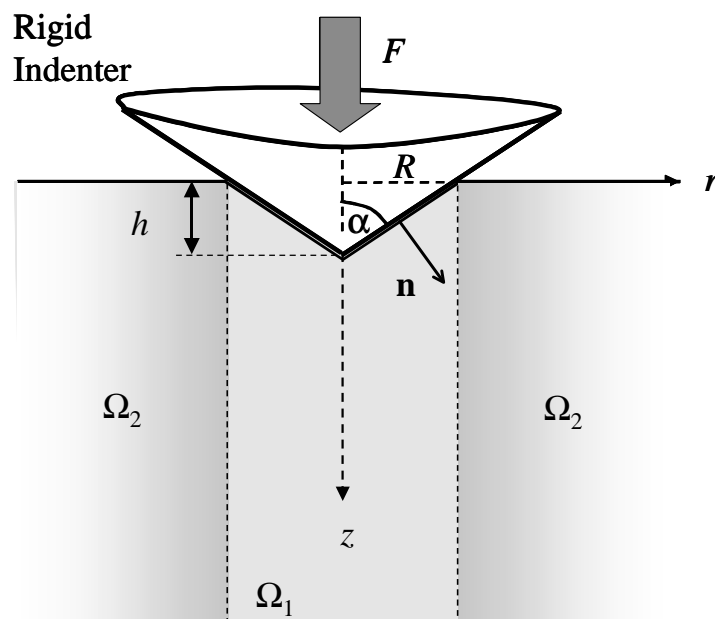
1. **Statically Admissible Stress Field**: For purpose of analysis, we separate the half-space Ω in two subdomains, noted respectively Ω_1 and Ω_2 (see figure). In these domains, we consider diagonal stress fields in a cylinder coordinate frame, which is of the form:

$$\boldsymbol{\sigma}' = \sigma'_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \sigma'_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma'_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \qquad (Q1)$$

   (a) For both domains, Ω_1 and Ω_2, specify precisely ALL conditions which statically admissible stress fields of the form (Q1) need to satisfy in Ω_1 and Ω_2. Hint: A special attention should be given to the frictionless contact condition at the indenter-material interface oriented by unit normal $\mathbf{n}$.

   (b) By assuming the stresses constants in Ω_1 and Ω_2, determine the expressions of the stresses $\sigma'_{rr}, \sigma'_{\theta\theta}, \sigma'_{zz}$ in both subdomains, that ensure that the stress field $\boldsymbol{\sigma}'$ is staticall admissible in $\Omega = \Omega_1 \cup \Omega_2$.

   (c) Display the stress field $\boldsymbol{\sigma}'$ for Ω_1 and Ω_2 in the Mohr Plane $(\sigma \times \tau)$. In both Mohr Plane and material plane, determine the surface and the corresponding stress vector, where the shear stress is maximum in Ω.

2. **Mohr-Coulomb Strength Criterion**: The material we consider is a Mohr-Coulomb material, for which the strength domain is defined by:

$$f(\boldsymbol{\sigma}) = \sigma_I(1 + \sin\varphi) - \sigma_{III}(1 - \sin\varphi) - 2c\cos\varphi \le 0$$

   where $\sigma_I \ge \sigma_{II} \ge \sigma_{III}$ are the principal stresses, and φ and c are the known friction angle, and the cohesion.

   (a) Display the Mohr-Coulomb criterion in the Mohr Plane $(\sigma \times \tau)$;

   (b) Show that the maximum value the hardness can take is given by:

$$H \le \max H = c\mathcal{F}(\varphi)$$

   Determine fuction $\mathcal{F}(\varphi)$ for the conical indentation test.

   (c) In the material plane, represent the orientation of the critical material surfaces, on which the Mohr-Coulomb criterion is reached.

   (d) DISCUSSION: Do you think that H/c determined with the diagonal stress fields Q1 is a lower or an upper bound of the actual H/c-ratio that one would expect to measure in an experiment?

1 Statically Admissible Stress Fields

A statically admissible stress field is a stress field which satisfies (i) the force boundary conditions, (ii) the stress vector continuity condition on any surface in the material; (iii) the symmetry of the stress tensor; (iv) the momentum balance.

For the conical nanoindentation test, the conditions are:

- In Ω_1:

  - Frictionless contact along indenter-material interface oriented by unit normal $\mathbf{n} = \cos\alpha\mathbf{e}_r + \sin\alpha\mathbf{e}_z$. The stress vector for a diagonal stress field reads:

$$\mathbf{T}(\mathbf{n}) = \begin{bmatrix} \sigma'_{rr} & & \\ & \sigma'_{\theta\theta} & \\ & & \sigma'_{zz} \end{bmatrix} \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} = \sigma'_{rr}\cos\alpha\mathbf{e}_r + \sigma'_{zz}\sin\alpha\mathbf{e}_z \tag{1}$$

    The frictionless contact implies that $\mathbf{t}\cdot\mathbf{T}(\mathbf{n}) = 0$, where $\mathbf{t} = \sin\alpha\mathbf{e}_r - \cos\alpha\mathbf{e}_z$, thus:

$$\begin{aligned} \mathbf{t}\cdot\mathbf{T}(\mathbf{n}) &= (\sin\alpha\mathbf{e}_r - \cos\alpha\mathbf{e}_z)(\sigma'_{rr}\cos\alpha\mathbf{e}_r + \sigma'_{zz}\sin\alpha\mathbf{e}_z) & (2) \\ &= \sigma'_{rr}\sin\alpha\cos\alpha - \sigma'_{zz}\cos\alpha\sin\alpha = 0 \Rightarrow \sigma'_{rr} = \sigma'_{zz} & (3) \end{aligned}$$

  - Force boundary condition on $z = 0$:

$$\mathbf{n} = -\mathbf{e}_z : \mathcal{N}^d = F\mathbf{e}_z \equiv -\int_{A=\pi R^2}\boldsymbol{\sigma}\cdot\mathbf{e}_z da = -HA\mathbf{e}_z; \ H = -\frac{1}{A}\int_{A=\pi R^2}\sigma'_{zz}da \tag{4}$$

  - Equilibrium:

$$\frac{\partial\sigma'_{rr}}{\partial r} = 0 \Longleftrightarrow \sigma'_{rr} = \sigma'_{\theta\theta}; \ \frac{\partial\sigma'_{\theta\theta}}{\partial\theta} = 0; \frac{\partial\sigma'_{zz}}{\partial z} = 0 \tag{5}$$

- In Ω_2:

  - zero stress boundary conditions:**

$$\text{on } z = 0; r > R; \mathbf{n} = -\mathbf{e}_z : \mathbf{T}(\mathbf{n} = -\mathbf{e}_z) = 0 \Rightarrow \sigma_{zz} = 0 \tag{6}$$

  - Equilibrium

$$\frac{\partial\sigma'_{rr}}{\partial r} = 0 \Longleftrightarrow \sigma'_{rr} = \sigma'_{\theta\theta}; \ \frac{\partial\sigma'_{\theta\theta}}{\partial\theta} = 0; \frac{\partial\sigma'_{zz}}{\partial z} = 0 \tag{7}$$

- Continuity Condition at $r = R$:

$$\text{on } z > 0; r = R; \sigma'^{(1)}_{rr} = \sigma'^{(2)}_{rr} \tag{8}$$

Thus, in summary from a combination of the previous equations:

$$\begin{aligned} \text{in } \Omega_1 \ : \ \boldsymbol{\sigma}' &= -H\left(\mathbf{e}_r\otimes\mathbf{e}_r + \mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_z\otimes\mathbf{e}_z\right) & (9) \\ \text{in } \Omega_2 \ : \ \boldsymbol{\sigma}' &= -H\left(\mathbf{e}_r\otimes\mathbf{e}_r + \mathbf{e}_\theta\otimes\mathbf{e}_\theta\right) & (10) \end{aligned}$$

The stress states are displayed in the Mohr-plane in the figure below. The hydrostatic stress field of domain 1 is represented as a point $(\sigma, \tau) = (-H, 0)$ in the Mohr-Plane. The Mohr-circle of domain 2, has a maximum value of $\sigma_I = 0$, and a radius of $H/2$. The surface where the shear stress is maximum is oriented by:

$$\mathbf{n} = \frac{\sqrt{2}}{2} \left(-\mathbf{e}_r + \mathbf{e}_z \right) \tag{11}$$

2 Mohr-Coulomb Strength Criterion

For a hydrostatic stress field, the Mohr-Coulomb criterion is always satisfied. Thus, the only restriction comes from domain 2, for which:

$$\sigma_I = 0; \sigma_{II} = \sigma_{III} = -H \tag{12}$$

Use in the Mohr-Coulomb strength criterion yields:

$$\frac{H}{c} \leq \mathcal{F}(\varphi) = \frac{2\cos\varphi}{1 - \sin\varphi} \tag{13}$$

The surface in domain Ω_2 on which the strength criterion is reached, is oriented by:

$$\mathbf{n} = -\sin\vartheta \mathbf{e}_r + \cos\vartheta \mathbf{e}_z; \ \vartheta(\mathbf{e}_z, \mathbf{n}) = \pi/4 - \varphi/2 \tag{14}$$

The solution is a lower bound, since it satisfies equilibrium and strength criterion. We should, however, note that the solution is not very realistic: it is independent of the apex angle, and does therefore not converge to the one of a flat indenter, i.e.:

$$Flat\ indenter : \frac{H}{c} \leq \mathcal{F}(\varphi) = \frac{4\cos\varphi}{(1 - \sin\varphi)^2} \tag{15}$$

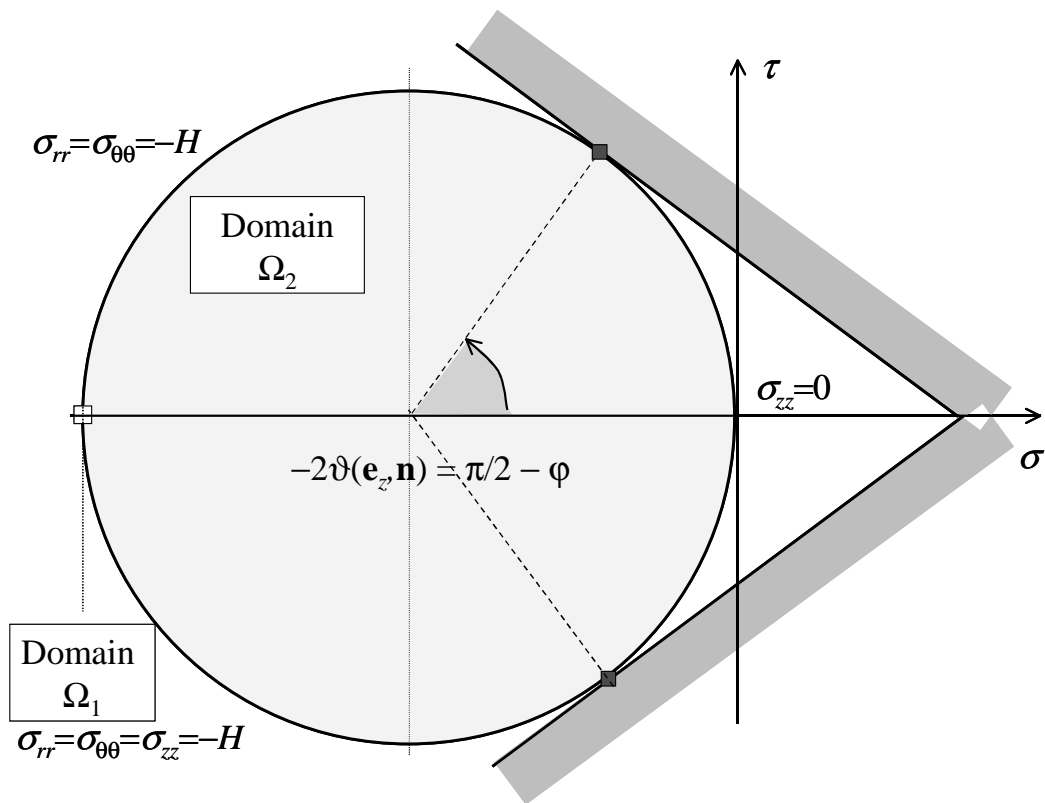Furthermore, it seems quite unrealistic that the stress state below the indenter is a hydrostatic stress field.

$\sigma_{rr}=\sigma_{\theta\theta}=-H$

Domain
Ω_2

$\sigma_{zz}=0$

$-2\vartheta(\mathbf{e}_z,\mathbf{n}) = \pi/2 - \varphi$

τ

σ

Domain
Ω_1

$\sigma_{rr}=\sigma_{\theta\theta}=\sigma_{zz}=-H$

Figure 1: