1.022 Introduction to Network Models **Fall 2018**

Homework 1

1.1 Graphs and Concepts

Graph properties ***[10 points]*** Consider the undirected graphs $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$, with the following sets of vertics / edges:

1. $\mathcal{V}_1 = \{1, \ldots, 6\}, \quad \mathcal{E}_1 = \{(1,2),(1,6),(2,3),(3,4),(4,5),(5,6)\}$
2. $\mathcal{V}_2 = \{1, \ldots, 8\}, \quad \mathcal{E}_2 = \{(2,1),(1,6),(1,5),(5,7),(3,5),(5,4),(4,8)\}$
3. $\mathcal{V}_3 = \{1, \ldots, 8\}, \quad \mathcal{E}_3 = \{(2,1),(1,3),(3,2),(5,8),(4,5),(6,4),(6,8),(4,7)\}$
4. $\mathcal{V}_4 = \{a,b,c,d,e,f,g\}, \quad \mathcal{E}_4 = \{(a,b),(b,c),(b,d),(c,d),(d,e),(e,f),(e,g),(f,g)\}$

For each graph provide all its properties from the following list. Note that a graph may possess none or more than one property: i) bipartite graph, ii) contains a cycle, iii) tree graph, iv) has a diameter $d < 4$, v) is not connected.

Graph properties II ***[10 points]*** Consider a graph $\mathcal{G}$ as described by the following adjacency matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the degree of node 1, the neighborhood of node 4, the diameter of the graph, and the longest cycle. Provide your reasoning (you may find it helpful to plot the graph).

Graph properties III ***[10 points]*** Let **A** be the adjacency matrix of an undirected network and **1** be the column vector whose elements are all 1. In terms of these quantities write expressions for:

1. the vector **k** whose elements are the degrees k_i of the vertices
2. the number m of edges in the network

Directed vs undirected graphs ***[10 points]*** Provide a counterexample for the following *wrong* statement: "Every weakly connected graph is strongly connected". Is the converse statement true? Provide a short proof by contradiction.

1.2 Graph properties and algebraic representations

[5 points] Consider the undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ defined by the following set of vertices

$$\mathcal{V} = \{a, b, c, d, e, f, g\},$$

and the following set of edges:

$$\mathcal{E} = \{(a, b), (b, c), (b, d), (c, d), (d, e), (e, f), (e, g), (f, g)\}.$$

Find two adjacency matrices describing the graph, and provide the labeling function $\ell : \mathcal{V} \to \mathbb{N}$ for the two matrices.

[5 points] Consider the adjacency matrix A of a generic undirected graph. What quantity does the matrix entry $[A^3]_{ij}$ count? Provide a short explanation. Hint: you might find it insightful to first consider the case $[A^2]_{ij}$, and to start by considering a simple graph like a path.

[5 points] Consider an undirected triangle graph. How many different ways are there to traverse the triangle via a cyclic path of length 3? Hint: in how many distinct ways can you arrange the node labels into a sequence of length 3?

[10 points] Prove the following: the number of triangles in an undirected graph is

$$\#\text{ triangles} = \frac{1}{6}\text{trace}A^3.$$

Hints: Recall that the trace of a matrix is just the sum of its entries on the main diagonal. From the previous exercises you should know what the entry A^3_{ii} counts. Ensure that you do not overcount.

[15 point] As we have seen above there can be different algebraic representations (in the form of adjacency matrices) of the same (unlabelled) graph. Prove that the formula for the number of triangles does not depend on the algebraic representation of the graph, i.e., the labelling of the nodes does not effect the formula.

Hints: Show that a permutation of the node labels according to a permutation matrix P results in a new adjacency matrix of the form $A' = PAP^T$. Argue with the properties of the trace that this means the above formula is invariant to relabelings of the nodes.

1.3 Triadic closure and link formation

[5 points] Explain the triadic closure hypothesis in 3 sentences.

[5 points] Given a graph as shown in Figure 1.1, with edges labelled as weak and strong. According to the theory of strong and weak ties, with the strong triadic closure assumption, how would you expect the unlabelled edge to be labeled? Give a brief (1-3 sentence) explanation for your answer.

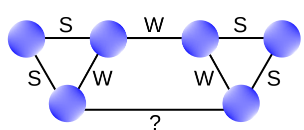

Figure 1.1: Network with strong and weak ties.