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Quiz 11

1. Consider the Hamiltonian matrix

$$\mathbf{H} = \frac{1}{3}\begin{pmatrix} 4 & 1 & 1 \\ 1 & 7 & -2 \\ 1 & -2 & 7 \end{pmatrix}$$

which has eigenvectors

$$6^{-1/2}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, 3^{-1/2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 2^{-1/2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

and eigenvalues 1, 2, and 3 (not necessarily in the same order as the eigenvectors).

A. Determine the one-to-one correspondence between eigenvectors and eigenvalues.

$$\frac{1}{3}\begin{pmatrix} 4 & 1 & 1 \\ 1 & 7 & -2 \\ 1 & -2 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} \qquad \text{eigenvalue 1}$$

$$\frac{1}{3}\begin{pmatrix} 4 & 1 & 1 \\ 1 & 7 & -2 \\ 1 & -2 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} \qquad \text{eigenvalue 2}$$

$$\frac{1}{3}\begin{pmatrix} 4 & 1 & 1 \\ 1 & 7 & -2 \\ 1 & -2 & 7 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix} \qquad \text{eigenvalue 3}$$

B. Construct, by assembling eigenvectors in the right way, the matrix $\mathbf{T}$ which you expect will diagonalize $\mathbf{H}$ in the sense $\mathbf{THT}^{\dagger}$ (but do not verify that it actually diagonalizes $\mathbf{H}$).

$$\text{T is either} \begin{pmatrix} \left(\frac{4}{6}\right)^{1/2} & 3^{-1/2} & 0 \\ -\left(\frac{1}{6}\right)^{1/2} & 3^{-1/2} & 2^{-1/2} \\ -\left(\frac{1}{6}\right)^{1/2} & 3^{-1/2} & -2^{-1/2} \end{pmatrix} \text{or its transpose.}$$

C. The time-evolution operator is: $\mathbf{U}(t,t_0) = \exp[-i\mathbf{H}(t-t_0)/\hbar]$. The matrix $\mathbf{U}(t,t_0)$, expressed in the same basis set of the original non-diagonal $\mathbf{H}$ is

$$\mathbf{U} = \mathbf{T}^\dagger \exp[-i\mathbf{THT}^\dagger(t-t_0)/\hbar]\mathbf{T}$$

where $\mathbf{THT}^\dagger$ is diagonal. Write the 3×3 diagonal matrix:

$\exp[-i\mathbf{THT}^\dagger(t-t_0)/\hbar] =$

$$\mathbf{THT}^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\mathbf{U} = \mathbf{T}^\dagger \begin{pmatrix} e^{-i(t-t_0)/\hbar} & 0 & 0 \\ 0 & e^{-i2(t-t_0)/\hbar} & 0 \\ 0 & 0 & e^{-i3(t-t_0)/\hbar} \end{pmatrix} \mathbf{T}$$