Lecture #6: Linear V(x). JWKB Approximation and Quantization

JWKB: Jeffreys, Wentzel, Kramers, Brillouin.

Last time: Normalization schemes for eigenfunctions which belong to continuously variable eigenvalues.

1. identities
2. $\psi_{\delta k}, \psi_{\delta p}, \psi_{\delta E}, \psi_{\text{box}}$: different normalization schemes
3. trick using box normalization $(\theta \text{ is } k, p, E)$

$$\left(\frac{\#\text{ states}}{\underset{\propto L}{\delta\theta}}\right)\left(\frac{\#\text{ particles}}{\underset{\propto 1/L}{\delta x}}\right) \text{ for box normalization}$$

4. $\frac{dn}{dE}$ ("density of states") often needed - alternate method via JWKB next lecture

1. $\boxed{\text{V(x) = } \alpha \text{ x linear potential}}$

   solve in momentum representation, $\phi(p)$, and take F.T. to $\psi(x) \rightarrow$ Airy functions
2. Semi-classical (JWKB) approx. for $\psi(x)$

* $p(x) = [(E - V(x))2m]^{1/2}$ Classical mechanical momentum function $\boxed{\text{dependence on } x.}$

* $\psi(x) = \underbrace{|p(x)|^{-1/2}}_{\text{envelope}} \exp\left[\pm \frac{i}{\hbar}\int_c^x p(x')dx'\right]$

* visualize $\psi(x)$ as plane wave with x-dependent wave vector

* useful for evaluating stationary phase integrals (localization, causality)

**** splicing across boundary between classical (E > V) and forbidden (E < V) regions] Next lecture

WKB Quantization Condition

$$\int_{x_-(E)}^{x_+(E)} p(x')dx' = \frac{h}{2}(n + 1/2)\; n = 0,1,\ldots$$

<u>Linear</u> Potential. V(x) = α x

$$\hat{H} = \frac{\hat{p}^2}{2m} + \alpha\hat{x}$$

coordinate representation

$$\hat{x} \to x$$
$$\hat{p} \to \frac{\hbar}{i}\frac{\partial}{\partial x}$$

momentum representation

$$\hat{p} \to p$$
$$\hat{x} \to i\hbar\frac{\partial}{\partial p}$$

(note $[\hat{x},\hat{p}] = i\hbar$ in both representations - prove this?)

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \alpha x$$

2nd order differential equation

$$\hat{H} = \frac{p^2}{2m} + i\hbar\alpha\frac{d}{dp}$$

1st order - much easier!

$$0 = (H - E)\phi(p)$$
$$0 = \left(\frac{p^2}{2m} + i\hbar\alpha\frac{d}{dp} - E\right)\phi(p)$$

Solve in <u>momentum representation</u> (a sometimes useful trick)

Schr. Eq. $\boxed{\frac{d\phi(p)}{dp} = -\frac{i}{\hbar\alpha}\left(E - p^2/2m\right)\phi(p)}$

Form of Solution $\phi(p) = Ne^{ap+bp^3}$

$\boxed{\frac{d\phi}{dp} \text{ gives } p^2 \text{ times } \phi(p)}$

$\boxed{\frac{d\phi}{dp} \text{ gives constant times } \phi(p)}$

when you take $\frac{d}{dp}$

Must solve for a and b

plug into Schr. Eq. and identify correspondences, term-by-term, to get

$$\boxed{\begin{aligned} a &= -\frac{iE}{\hbar\alpha} \\ b &= \frac{i}{6\hbar\alpha m} \end{aligned}}$$

$$\boxed{\phi(p) = N\exp\left[-\frac{i}{\hbar\alpha}\left(Ep - p^3/6m\right)\right]}$$

easy? Note that, if p is real, ϕ(p) is oscillatory

$$\phi^*(p)\phi(p) = 1! \qquad \therefore \quad N = 1!$$

Now p is an observable, so it must be real. Thus $\phi(p)$ is defined for all (real) p and is oscillatory in p for all p. $\phi(p)$ is NEVER exponentially increasing or decreasing if p is real!

IT IS STRANGE THAT $\phi(p)$ does not distinguish between classically allowed and forbidden regions. IS THIS REALLY STRANGE? If we allow p to be imaginary in order to deal with classically forbidden regions, $\phi(p)$ becomes an increasing or decreasing exponential. When we extend the solution to the Schrödinger equation into the classically forbidden region, p is imaginary and $\phi(p)$ is exponentially increasing or decreasing.

If we insist on working in the $\psi(x)$ picture, we must perform a Fourier Transform.

$$\psi(x) = N'\int_{-\infty}^{\infty} e^{ipx/\hbar}\phi(p)dp$$

$$\psi(x) = N'\int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar\alpha}\left\{\underbrace{p(\alpha x - E) + p^3/6m}_{\text{odd function of p: O(p)}}\right\}\right]dp$$

$$e^{i\theta} = \underbrace{\cos\theta}_{even} + i\underbrace{\sin\theta}_{odd}$$

$$\int_{-\infty}^{\infty} \sin O(p)dp = 0$$ since sin $O(p)$ is odd wrt p → –p.

$$\boxed{\psi(x) = N'\int_{-\infty}^{\infty} \cos\left[\frac{(\alpha x - E)p + p^3/6m}{\hbar\alpha}\right]dp}$$ Solution!

$$Ai(z) = \pi^{-1/2}\int_0^{\infty} \cos\left(s^3/3 + sz\right)ds$$

$$\boxed{\begin{aligned} E = V(x) &= \alpha x_{tp} \\ x_{tp} &= E/\alpha \end{aligned}}$$

Surprise! This is a *named* (Airy) function and a *tabulated* integral

* numerical tables for x near turning point i.e., $x \approx E/\alpha$
* analytic "asymptotic" functions for x far from turning point.

  useful for deriving energy levels as an explicit function of quantum numbers and for matching wave functions across boundaries.

* zeroes of Airy functions [$Ai(z_i)=0$] and of derivatives of Airy functions [$Ai'(z'_i)=0$] are tabulated. (Useful for matching across center symmetry-point of potentials with definite even or odd symmetry.) [Two kinds of Airy functions, Ai and Bi.]

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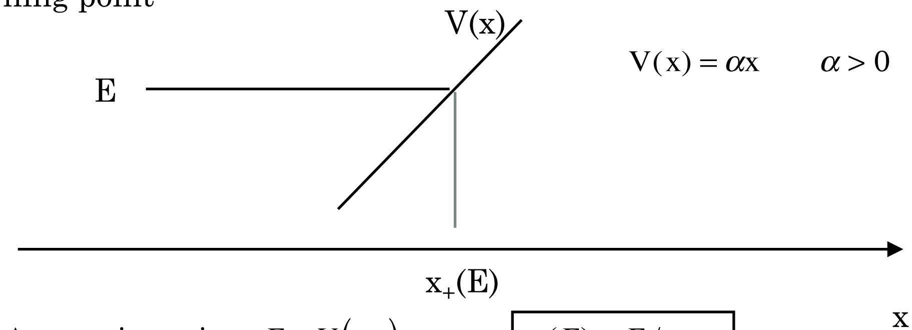
$$\boxed{\mathrm{Ai}(z) = \pi^{-1/2}\int_0^\infty \cos\left(\frac{s^3}{3} + sz\right) ds}$$

for our specific problem
$$s \equiv p(2m\hbar\alpha)^{-1/3} \qquad (if\ \alpha > 0)$$
$$z \equiv \frac{(\alpha x - E)}{\alpha}\left[2m\alpha/\hbar^2\right]^{1/3}$$

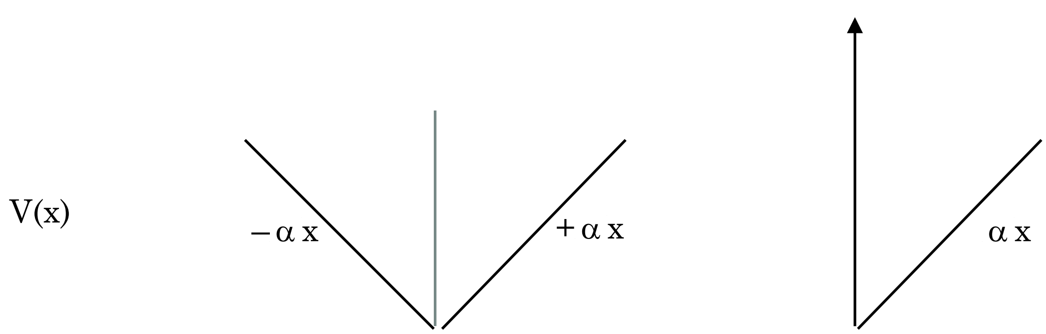
Turning point

$V(x) = \alpha x \qquad \alpha > 0$



At a turning point $E = V\left(x_+\right) = \alpha x_+ \therefore \boxed{x_+(E) = E/\alpha}$

Problems with linear potentials: boundary conditions

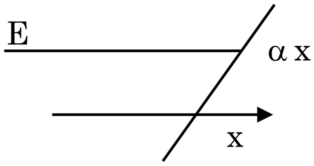


When there is symmetry (or 1/2 symmetry) we need to know the locations of the zeroes of $\underbrace{d\psi/dx}_{\text{for even functions}}$ and $\underbrace{\psi(x)}_{\substack{\text{for odd functions}\\ \text{or } \infty \text{ barrier}}}$.

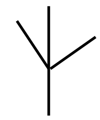
tables of zeroes of
Ai (z) and Ai′(z)
"z_n" "z'_n"

See Handout

When there is no symmetry, must match or join Ai (or, more precisely, a linear combination of Ai and Bi) and Ai′ across boundaries, but we do not need to actually look at the Airy function itself near the joining point.



This is not as bad as it seems because we are usually far from the *turning* point at an internal *joining* point and can use *analytic asymptotic expressions* for Ai(z).



2 linear potentials of different |slope|.

For $\alpha > 0$ there are 2 cases (classical and non-classical regions)

(i) $z \ll 0, E > V(x)$ classically allowed region

$$\mathrm{Ai}(z) \to \pi^{-1/2}(\underbrace{-z}_{positive})^{-1/4}\sin\left[\frac{2}{3}(\underbrace{-z}_{\text{x is in here}})^{3/2} + \underbrace{\pi/4}_{\text{phase shift}}\right]$$

asymptotic form for $z \ll 0$.

* oscillatory, but wave vector, k, varies with x

* Ai vanishes as $x \to -\infty$ because of $(-z)^{-1/4}$ factor

* Bi is needed for case where Airy function must vanish as $x \to +\infty$ in classical region

(ii) well behaved

$z >> 0$, $E < V(x)$ forbidden region

$$\mathrm{Ai}(z) \to \left(\pi^{-1/2}/2\right)\underbrace{z^{-1/4}}_{\text{positive}}\underbrace{e^{-(2/3)z^{3/2}}}_{\text{decreasing exponential}}$$

asymptotic form for $z \gg 0$.

* not oscillatory, monotonic

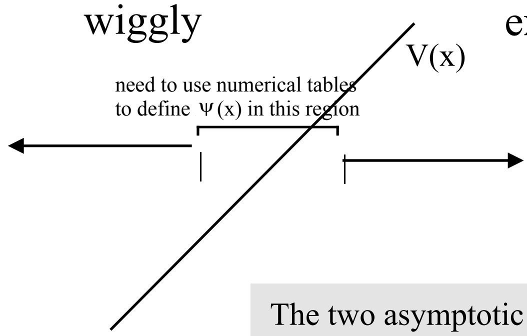
* Ai vanishes as $x \to +\infty$

* Bi vanishes as $x \to -\infty$ in forbidden region

Cartoon

damped wiggly

damped exponential



The two asymptotic forms of Ai(z) are not normalized, but their <u>amplitudes</u> (& phases) can and must be matched. This links boundary condition at $x \to +\infty$ to boundary condition at $x \to -\infty$.

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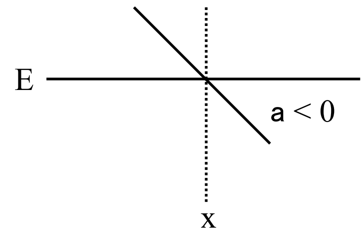
NonLecture

OTHER CASE: $\alpha < 0 \rightarrow z \equiv -\frac{(|\alpha| x + E)}{|\alpha|}\left[2m|\alpha|/\hbar^2\right]^{1/3}$



for this case, we need Bi(z) instead of Ai(z)

$$Bi(z) \rightarrow \left(\pi^{-1/2}/2\right)|z|^{-1/4}\exp\left[-\frac{2}{3}|z|^{3/2}\right] \qquad \text{(forbidden region, } z \ll 0.)$$

$$Bi(z) \rightarrow \pi^{-1/2}|z|^{-1/4}\cos\left[\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right] \qquad \text{(allowed region, } z \gg 0.)$$

What is so great about V(x) = α x? Ψ(x) seems ugly — need lookup tables, complicated solutions!

But Ai(z) turns out to be key to generalization of quantization of *all* (well behaved) V(x)!

These are semi-classical JWKB Ψ(x) functions — They blow up near turning points (i.e. on both sides). The Ai(z)'s permit matching of JWKB Ψ(x)s across the large gap where Ψ_{JWKB} is invalid, ill-defined.

(**J**EFFREYS)

WENTZEL

KRAMERS

BRILLOUIN

JWKB provides a way to get $\Psi_n(x)$ and E_n without solving differential equations or performing a Fourier Transform.

But actually, the differential equations are easy to solve numerically. The reason we care about JWKB is that it provides a basis for:

* physical interpretation (semi-classical)
* RKR inversion from $E_{vJ} \rightarrow V_J(R)$. [Rydberg, Klein, Rees]
* semi-classical quantization.
* the link to classical mechanics is essential for wavepacket pictures.

(generalize on e^{ikx} for free particle by letting $k = p(x)/\hbar$ depend explicitly on x (why does this not violate $[x,p]=i\hbar$?)

$$\psi_{\text{JWKB}} = \underbrace{|p(x)|^{-1/2}}_{\text{classical envelope}} \exp\left[\pm\frac{i}{\hbar}\int_c^x p(x')dx'\right]$$

$$p(x) = [2m(E - V(x))]^{1/2}$$

No violation because k(x) and p(x) are classical mechanical functions of x, not QM operators.

phase factor: choose c to satisfy boundary conditions

p(x) is pure real (classically allowed) or pure imaginary (classically forbidden). p(x) is not the Q.M. momentum. It is a classically motivated function of x, which has the form of the classical mechanical momentum and has the property that the $\lambda = \frac{h}{p}$ varies with x in a reasonable way.

* $|p(x)|^{-1/2}$ is probability amplitude envelope because

probability $\propto \frac{1}{v}$ so amplitude $\propto \sqrt{\frac{1}{v}}$ (v is velocity)

* $\exp-\left[\frac{i}{\hbar}\int_c^x p(x')dx'\right]$ is the generalization of $e^{ipx/\hbar}$ to non-constant V(x).

*node spacing $\frac{\lambda(x)}{2} = \frac{h}{2p(x)}$

**gives* easily identifiable stationary phase region for many wiggly integrands.

(Both ψ's have same λ at stationary phase point $x_{s.p.}$)

Long Nonlecture derivation/motivation of the JWKB splice across the turning point, even though the JWKB functions are not valid near the turning point.

Try $\psi(x) = N(x)\exp\left[\pm\frac{i}{\hbar}\int_c^x p(x')dx'\right]$

plug into Schr. Eq. and get a new differential equation that N(x) must satisfy

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\psi = 0$$

$$\frac{d^2\psi}{dx^2} + \frac{1}{\hbar^2}p(x)^2\psi = 0 \qquad **$$

* derived in box below

$$0 = \left[N'' \pm \frac{2ip(x)}{\hbar}N' \pm \frac{ip'(x)}{\hbar}N\right]\exp\left[\pm\frac{i}{\hbar}\int_c^x p(x')dx'\right]$$

This is a new Schr. Eq. for N(x). Now make an approximation, to be tested later, that N'' is negligible everywhere. This is based on the expectation that a slowly varying V(x) will lead to a slowly varying N(x).

*

$$\frac{d\psi}{dx} = \left[N' \pm \frac{i}{\hbar}p(x)\right]\exp\left[\pm\frac{i}{\hbar}\int_c^x p(x')dx'\right]$$

$$\frac{d^2\psi}{dx^2} = \left[N'' \pm \frac{i}{\hbar}N'p \pm \frac{i}{\hbar}Np' \pm \frac{ip}{\hbar}\left(N' \pm \frac{i}{\hbar}Np\right)\right]\exp\left[\pm\frac{i}{\hbar}\int_c^x p(x')dx'\right]$$

$$= \left[N'' \pm \frac{2i}{\hbar}N'p \pm \frac{ip'}{\hbar}N - \frac{p^2}{\hbar^2}N\right]\exp\left[\pm\frac{i}{\hbar}\int_c^x p(x')dx'\right]$$

$$0 = \frac{d^2\psi}{dx^2} + \frac{p^2}{\hbar^2}N\exp\left[\pm\frac{i}{\hbar}\int_c^x p(x')dx'\right]$$

$$0 = \left[N'' \pm \frac{2ip(x)}{\hbar}N' \pm \frac{ip'}{\hbar}N\right]\exp\left[\pm\frac{i}{\hbar}\int_c^x p(x')dx'\right]$$

so, if we neglect N″, we get for the first term in [ ]

$2pN' + p'N = 0$

$$\text{if } p \neq 0, \text{ then } 2p^{1/2}\left[p^{1/2}N' + \frac{1}{2}p^{-1/2}p'N\right] = 0$$

$$\frac{d\left(Np^{1/2}\right)}{dx} = \left[N'p^{1/2} + \frac{1}{2}p^{-1/2}p'N\right]$$

$$\therefore \frac{d\left(Np^{1/2}\right)}{dx} = 0$$

$$N(x)p^{1/2}(x) = \text{constant}$$

$$\boxed{\therefore\ N(x) = cp(x)^{-1/2}}$$

OK, now we have a form for N(x) that we can use to tell us what conditions must be satisfied so that $N''(x)$ is negligible everywhere.

$$N = cp^{-1/2}$$

$$\frac{dp^{-1/2}}{dx} = -\frac{1}{2}p^{-3/2}\frac{dp}{dx} \qquad p(x) = \left[2m\left(E - V(x)\right)\right]^{1/2}$$

$$\frac{dp}{dx} = \frac{1}{2}\left[2m\left(E - V(x)\right)\right]^{-1/2}(-2m)\frac{dV}{dx}$$

$$= \frac{1}{2}p^{-1}(-2m)\frac{dV}{dx} = -mp^{-1}\frac{dV}{dx}$$

$$\therefore \frac{dp^{-1/2}}{dx} = p^{-5/2}\frac{m}{2}\frac{dV}{dx}$$

$$\frac{d^2p^{-1/2}}{dx^2} = \frac{m}{2}\frac{dV}{dx}\left(-\frac{5}{2}\right)p^{-7/2}\left[-\frac{m}{p}\frac{dV}{dx}\right] + p^{-5/2}\frac{m}{2}\underbrace{\frac{d^2V}{dx^2}}_{\text{ignore}}$$

$$\therefore \boxed{N'' = c\frac{5}{4}m^2 p^{-9/2}\left(\frac{dV}{dx}\right)^2}$$

But we have made several assumptions about N'':

$$*\left|N''\right| \ll \left|\frac{2ip}{\hbar}N'\right| = \left|+\frac{icm}{\hbar}p^{-3/2}\frac{dV}{dx}\right|$$

$$*\left|N''\right| \ll \left|\frac{ip'}{\hbar}N\right| = \left|-\frac{icm}{\hbar}p^{-3/2}\frac{dV}{dx}\right|$$

$$*\left|N''\right| \ll \frac{p^2}{\hbar^2}N = \frac{c}{\hbar^2}p^{+3/2}$$

all of this is satisfied if

$$\left|\frac{5}{4}\frac{m\hbar}{i}\left(\frac{dV}{dx}\right)p^{-3}\right| \ll 1$$

Is this the JWKB validity condition? If it is, what does it mean?

Spirit of JWKB: if initial JWKB approximation is not sufficiently accurate, iterate:

$p(x) \to \psi_0(x)$ (ordinary JWKB)

$\psi_0(x) \to p_1(x)$

$p_1(x) \to \psi_1(x)$ (first order JWKB)

$$e.g.\quad \frac{d^2\psi_0}{dx^2} + \frac{p_1^2}{\hbar^2}\psi_0 = 0 \to p_1(x) = \left[-\frac{\hbar^2}{\psi_0(x)}\frac{d^2\psi_0}{dx^2}\right]^{1/2}$$

see ** Eq. on p. 6-8

$$\psi_1(x) = \left|p_1(x)\right|^{-1/2}\exp\left[\pm\frac{i}{\hbar}\int_c^x p_1(x')dx'\right]$$

iterative improvement of accuracy

$p_1(x)$ is not smaller than $p_0(x)$, but it has more nearly correct wiggles in it.

END OF NONLECTURE

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Resume Lecture

$$\psi(x) \approx \underbrace{|p(x)|}_{\text{envelope}}^{-1/2} \exp\left[\pm \frac{i}{\hbar}\int_c^x p(x')\,dx'\right]$$

c ← adjustable phase shift.

provided that $\frac{d^2V}{dx^2}$ is negligible

AND

$$\underbrace{\frac{\hbar m}{|p|^3}\frac{dV}{dx}}_{\substack{\text{required for } N''(x) \\ \text{to be negligible}}} \ll 1 \left(\text{satisfied by } \lambda(x)\left|\frac{dp}{dx}\right| < |p(x)| \text{ or } \frac{d\lambda}{dx} \ll 1\right)$$

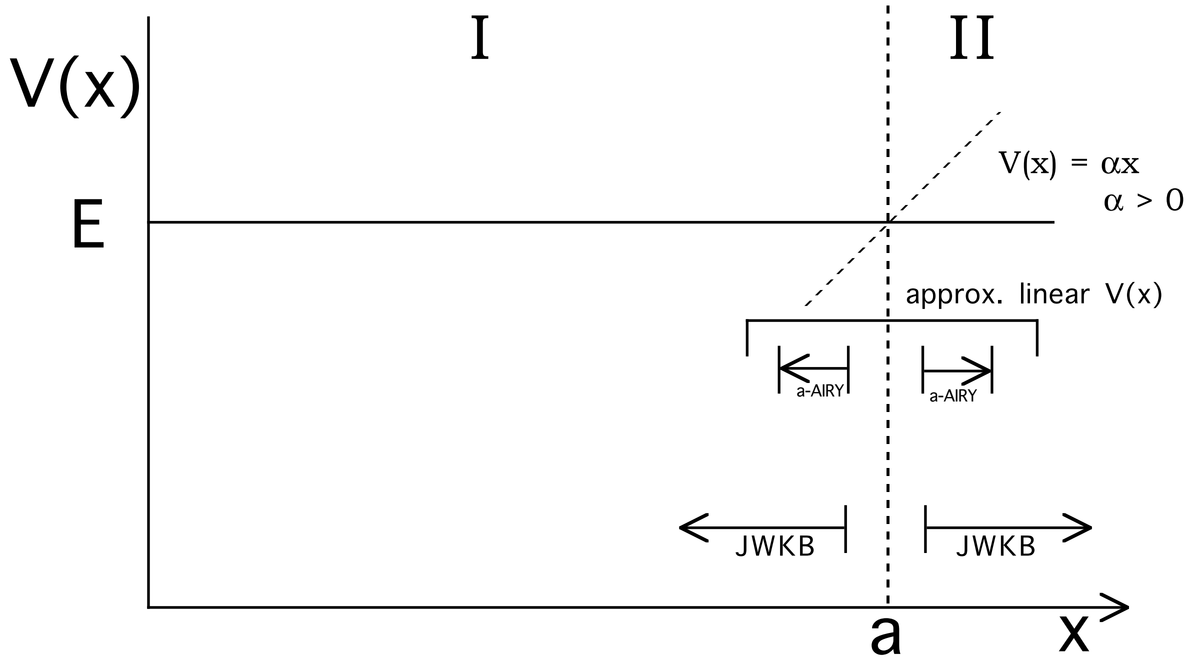
Next need to work out connection of $\Psi_{JWKB}(x)$ functions across region of x where the JWKB approx. breaks down (at turning points!).

$$\left|\frac{d\lambda}{dx}\right| \to \infty \text{ at turning point because } p(x) \to 0 \qquad \text{Looks BAD!}$$

BUT ALL IS NOT LOST — near enough to a turning point <u>all potentials</u> V(x) look like V(x)= α x! We have Airy functions that are solutions to the Schrödinger Equation for this linear potential.

Now our job is to show that <u>asymptotic – AIRY</u> and <u>JWKB</u> are identical for a small region not too close and not too far on both sides of each turning point.

THIS PERMITS ACCURATE SPLICING OF Ψ(x) ACROSS TURNING POINT REGION!



First use Airy to splice across I,II junction

<u>Region I</u> $E > V(x)$ classical

$$\psi^{I}_{a-AIRY} \sim \pi^{-1/2}(-z)^{-1/4}\sin\left[\frac{2}{3}(-z)^{3/2}+\pi/4\right]$$

$$z=\frac{(\alpha x-E)}{\alpha}\left[\frac{2m\alpha}{\hbar^2}\right]^{1/3}$$

at turning point $E = V(a) = \alpha a$ so $\left[\frac{\alpha x - E}{\alpha}\right] = (x-a)$

$$\boxed{z=(x-a)\left(\frac{2m\alpha}{\hbar^2}\right)^{1/3} \ll 0 \qquad \text{when } x \ll a}$$

Region I/II splice using a-Airy.

<u>Region II</u> $E < V(x)$ forbidden region, $z \gg 0$

$$\psi^{II}_{a-AIRY} \sim \frac{\pi^{-1/2}}{2} z^{-1/4} e^{-(2/3)z^{3/2}}$$

Now consider ψ_{JWKB} for a linear potential and show that it is identical to a-Airy!

$$\psi_{JWKB} \sim c_{\pm}|p(x)|^{-1/2}\exp\left[\pm\frac{i}{\hbar}\int_a^x p(x')dx'\right]$$

Then use WKB

both c_+ and $c\_$ additive terms could be present

$$p(x) \equiv [2m(E-V(x))]^{1/2}$$

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$x < a$	classical ,	p is real ,	ψ_{JWKM}	oscillates
$x > a$	forbidden ,	p is imaginary ,	ψ_{JWKB}	is exponential

pretend V(x) looks linear near x = a (ℓ-JWKB)

linear

$$p(x) = [2m\alpha(a-x)]^{1/2}$$

$$\int_a^x p(x')dx' = (2m\alpha)^{1/2}\int_a^x (a-x')^{1/2}dx'$$

$$= (2m\alpha)^{1/2}\left(-\frac{2}{3}\right)(a-x')^{3/2}\Big|_a^x$$

$$= -(2m\alpha)^{1/2}\frac{2}{3}(a-x)^{3/2}$$

Region I

$$\psi^I_{l-JWKB}(x) \sim |p(x)|^{-1/2}\left[Ae^{i\theta}+Be^{-i\theta}\right]$$

$$= |p(x)|^{-1/2}C\sin(\theta+\phi)$$

Define the JWKB phase factor, θ(x):

$$\theta = \frac{1}{\hbar}\int_a^x p(x')dx' = -\left(\frac{2m\alpha}{\hbar^2}\right)^{1/2}\frac{2}{3}(a-x)^{3/2}$$

Now compare θ(x) to z(x)

but, earlier , $z = (x-a)\left(\frac{2m\alpha}{\hbar^2}\right)^{1/3}$ $\quad \therefore \theta = -\frac{2}{3}(-z)^{3/2}$

$$\therefore p = (2m\alpha\hbar)^{1/3}(-z)^{1/2}$$ for exponential factor

$$|p|^{-1/2} = (2m\alpha\hbar)^{-1/6}(-z)^{-1/4}$$ for pre-exponential factor

Thus, putting all of the pieces together

$$\psi^{\mathrm{I}}_{\ell-\mathrm{JWKB}} = \overbrace{-(2m\alpha\hbar)^{-1/6}(-z)^{-1/4}}^{-|p|^{-1/2}} C \ \sin\left[\overbrace{\frac{2}{3}(-z)^{3/2}}^{-\theta} - \phi\right]$$

$$= \psi^{\mathrm{I}}_{a-\mathrm{AIRY}} \qquad \text{If } C = -(2m\alpha\hbar)^{1/6}\pi^{-1/2}$$

$$\phi = -\pi/4$$

$\psi^{\mathrm{I}}_{\ell-\mathrm{JWKB}}$ exactly splices onto $\psi^{\mathrm{I}}_{a-\mathrm{AIRY}}$

with a $\boxed{\pi/4 \text{ phase factor}}$ (shifted from what the argument of sine would have been if one had started the phase integral at x = a

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Similar result in Region II

$$\psi^{\mathrm{II}}_{\mathrm{JWKB}} \sim Ae^{-f(x)} + Be^{+f(x)}$$

at $x \to +\infty \qquad f(x) \to \infty \qquad \therefore B = 0$

$$\therefore \psi^{\mathrm{II}}_{\ell-\mathrm{JWKB}} = A(2m\alpha)^{-1/4}(x-a)^{-1/4}\exp\left[-\left(\frac{2m\alpha}{\hbar^2}\right)^{1/2}\frac{2}{3}(x-a)^{3/2}\right]$$

which is equal to $\psi^{\mathrm{II}}_{a-\mathrm{AIRY}}$ if $A = (2m\alpha\hbar)^{+1/6}\pi^{-1/2}/2$

$$\text{Final step:} \quad \psi^{\mathrm{I}}_{\mathrm{JWKB}} \leftrightarrow \psi^{\mathrm{I}}_{a-\mathrm{AIRY}}\ , \ \psi^{\mathrm{II}}_{\mathrm{JWKB}} \leftrightarrow \psi^{\mathrm{II}}_{a-\mathrm{AIRY}}$$

require $A = -C/2$

perfect match on opposite sides of turning point.

Ai(z) is valid in region where ψ_{JWKB} is invalid.

The logic is complicated, but the analysis assures that matching of $\Psi^{\mathrm{I}}_{\mathrm{JWKB}}$ to $\Psi^{\mathrm{II}}_{\mathrm{JWKB}}$ is valid and that one gets an extra π/4 phase factor at each turning point in the classically allowed region. This corresponds to the extra phase accumulated in the non-classical region so that $\Psi(\pm\infty) \to 0$. The energy levels are lowered below where they would have been if the wavefunctions in the classically allowed region were zero at turning points.