5.73 Problem Set 3

Due Friday, Oct. 7

All of these problems concern a particle in a delta function potential:

$$V(q) = V_0 \delta(q)$$

where V_0 is a constant .Note that the potential is attractive if $V_0 < 0$ and repulsive if $V_0 > 0$. We want to describe a particle in this potential in both the attractive and repulsive cases.

1. First, let us look at the positive energy eigenstates of the delta function potential (repulsive or attractive). Just as in the case of the step potential in class, we can write down the state of the system either to the left or right of the potential:

$$|\psi\rangle = \begin{cases} |\psi_L\rangle = A_L|+p\rangle + A_L|-p\rangle & \text{on the left} \\ |\psi_R\rangle = A_R|+p\rangle + B_R|-p\rangle & \text{on the right} \end{cases}$$

We need two equations to fix the eigenstate. We obtain one by requiring the two wavefunctions to take on the same value at $q=0$: $\psi_L(0) = \psi_R(0)$. However, show that $\dfrac{d\psi_L}{dq}(0) \neq \dfrac{d\psi_R}{dq}(0)$ so that the derivative of the wavefunction is discontinuous at the origin. Determine a formula for $\dfrac{d\psi_L}{dq}(0) - \dfrac{d\psi_R}{dq}(0)$ in terms of m, V_0 and $\psi(0)$. [Hint: Integrate the eigenvalue equation $\langle q|\hat{H}|\psi\rangle = E\langle q|\psi\rangle$ from $+\varepsilon$ to $-\varepsilon$ and take the limit as $\varepsilon \to 0$.] What physical justification can we give for this discontinuity?

2. Now, let's consider the **repulsive** case in particular. Our experiment involves particles that are incident from the right on the barrier with energy $E > 0$.

   a. What are the proper boundary conditions on $|\psi\rangle$ in this case? What are the possible values of p far from the barrier?

   b. Determine the transmission, T, and reflection, R, coefficients for this barrier as a function of E. Be careful to account for the discontinuity in $\dfrac{d\psi}{dq}$ at the origin. You

may wish to convert to natural units where $m = V_0 = \hbar = 1$, in which case the unit of energy is $E_0 = \dfrac{mV_0{}^2}{\hbar^2}$. Is there any structure in the transmission or reflection that might be useful in characterizing the strength of the barrier? What happens in the low energy and high energy limits? Please feel free to use plots to illustrate your conclusions.

3. Now, let us consider scattering off an **attractive** delta function potential with the same experimental set-up as in the previous problem (particles incident from the right with energy $E > 0$).
    a. Determine the transmission and reflection coefficients in this case. Is there any structure in $T(E)$ or $R(E)$ that might help identify the strength of the potential? Are there any qualitative differences between the repulsive and attractive cases?
    b. Just for fun, let us pretend we can inject particles with negative energies ($E < 0$). Show that, if we just continue to use the expressions from part a., something very peculiar happens for $E = -\tfrac{1}{2}$. What might cause this?