

Biot Numbers by David Adrian

This document is a review of some of the concepts of heat and mass transfer, particularly focusing on the dynamics at the interface between two disparate materials, such as the boundary of a solid particle submerged in a fluid. In the cases considered, the interface is stationary and there is no phase change or chemical reaction at the interface.*

The Biot number is a dimensionless group that compares the relative transport resistances, external and internal. It arises when formulating and non-dimensionalizing the boundary conditions for the typical conservation of species/energy equation for heat/mass transfer problems.

If your problem consists of an object suspended in a well mixed fluid, commonly you only need to calculate the dynamics of the object (such as the temperature as a function of position and time). If we focus on the fluid/object interface, the convective flux from the bulk fluid to the object must equal the diffusive flux from the surface to the interior of the object. This is typically formulated as a Robin boundary condition at the interface. For example, consider the unsteady heat transfer in a solid sphere at initial temperature T_0 submerged in a fluid of temperature T_∞ (this is also the “bulk” temperature, and could be given the symbol T_b).

At the fluid-solid interface, the flux of heat into the sphere from the fluid must equal the flux of heat from the surface of the sphere to the interior.

$$\begin{aligned}\bar{q}_{\text{exterior}} &= \bar{q}_{\text{interior}} \\ \bar{q}_{\text{exterior}} &= h(T_s - T_\infty)\bar{n} \\ \bar{q}_{\text{interior}} &= -k_T \nabla T \Big|_{\text{surface}}\end{aligned}$$

The variables are defined as follows: q is the heat flux, h is the heat transfer coefficient in the fluid, T is the temperature, \bar{n} is the outwardly pointing normal from the solid, and k_T is the thermal conductivity of the solid. (Also recall that the heat transfer coefficient can be obtained from correlations, and is basically just $k_{T,\text{fluid}} / \delta_T$ in systems without any interfacial reactions or phase changes. The variable δ_T is the thickness of the thermal boundary layer.)

Since our system is spherically symmetric:

$$\begin{aligned}\bar{q}_{\text{exterior}} &= h(T_s - T_\infty)\bar{e}_r \\ \bar{q}_{\text{interior}} &= -k_T \frac{dT}{dr} \Big|_{\text{surface}} \bar{e}_r \\ -k_T \frac{dT}{dr} \Big|_{\text{surface}} &= h(T_s - T_\infty)\end{aligned}$$

* I hope it is a review! TAs should be knowledgeable about this stuff if you have questions.

We may non-dimensionalize the system by changing variables. We must pick a characteristic length scale and temperature scale with the hope that our dimensionless temperature scale will be $O(1)$ [†], i.e. it goes from 0 to 1. Also, we want the derivatives of that function with respect to our dimensionless length to also be $O(1)$. This is getting a good scaling for the problem.

In this problem, a good dimensionless temperature is:

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}$$

This temperature variable starts out at 1 and decays to zero once the solid and fluid temperatures are equal.

A good dimensionless length is the inverse of the surface area to volume ratio:

$$\eta = r / L = r \frac{SA}{V} = r \frac{4\pi R^2}{(4/3)\pi R^3} = \frac{3r}{R}$$

$L = R/3 = O(R)$ in the case of a sphere so “ $L = R$ ” would also be an okay choice. **L is the “typical” length scale that heat in the solid particle must diffuse to get to the surface.**

Using our definitions of dimensionless length and temperature, our Robin boundary condition becomes:

$$-k_T \frac{(T_0 - T_\infty)}{L} \frac{d\theta}{d\eta} \Big|_{surface} = h(T_s - T_\infty)$$

$$-\frac{d\theta}{d\eta} \Big|_{surface} = \frac{hL}{k_T} \theta \Big|_{surface} = N_{Bi} \theta \Big|_{surface}$$

$$N_{Bi} = \frac{hL}{k_T} = \frac{L/k_T}{1/h} = \frac{\text{"internal diffusion resistance"}}{\text{"external convection resistance"}}$$

By definition, our dimensionless temperature is at most 1. Consider the effect of Biot number on the problem.

Case 1: $N_{Bi} \ll 1$

$$-\frac{d\theta}{d\eta} \Big|_{surface} = N_{Bi} \theta \Big|_{surface}$$

Since $\theta \Big|_{surface}$ is at most one, in order for the equation to be true the surface gradient in our dimensionless temperature is also small (as small as the Biot number). This means that a good approximation to the dynamics can be found from a uniform temperature throughout the sphere (a lumped system model). This means that the “external convection resistance” dominates the problem, and the “internal diffusion resistance” is

[†] “Big O Notation” is used to state in rough terms the magnitude of terms relative to each other, usually only considering the order of magnitude or scaling

small relative to the convection resistance so that it can be neglected when determining the total resistance. (Think back to the “resistors in series” analogy in heat transfer.)

For example, a small heat transfer Biot numbers can arise in the case of a small conductive metal sphere in a stagnant fluid such as air. The convection speed is very slow (the resistance is large and h is small because the fluid is stagnant) and the conduction speed is very high (the thermal resistance is small, or the thermal conductivity is large, or the distance that heat has to diffuse in the object is very small).

Case 2: $N_{Bi} \gg 1$

On the other hand, if our Biot number is very large, the gradient must either be very large at the surface or $\theta|_{surface}$ must be very small. However, since we scaled the derivative

properly, $\frac{d\theta}{d\eta}|_{surface}$ should be $O(1)$. This means that $\theta|_{surface}$ is very small. Recall that if θ

is zero, this corresponds to the surface being in equilibrium with the bulk temperature (it takes the bulk value). This simplifies our boundary condition for the problem so that we can just use the condition that $T(r = R) = T_s = T_\infty$ and still get good results.

Mass Transfer Biot Numbers

In the case of mass transfer, the definition of the Biot number could get a little more complicated because the partition coefficient between phases is involved. In this class, however, we aren't worrying about partition coefficients, probably because we can't easily measure the internal concentrations anyway so we lump their effect into other unknown constants (such as the surface reaction rate constant).

The analogous case of a catalyst particle in a reactor fluid gives us the Robin boundary condition:

$$\vec{W}_{\text{exterior}} = k_m (c_s - c_\infty) \vec{e}_r$$

$$\vec{W}_{\text{interior}} = -D_e \left. \frac{dc}{dr} \right|_{\text{surface}} \vec{e}_r$$

$$-D_e \left. \frac{dc}{dr} \right|_{\text{surface}} = k_m (c_s - c_\infty)$$

$$\theta = \frac{c - c_\infty}{c_0 - c_\infty}$$

$$\eta = r/L = r \frac{SA}{V} = r \frac{4\pi R^2}{(4/3)\pi R^3} = \frac{3r}{R}$$

$$-D_e \frac{(c_0 - c_\infty)}{L} \left. \frac{d\theta}{d\eta} \right|_{\text{surface}} = k_m (c_s - c_\infty)$$

$$-\left. \frac{d\theta}{d\eta} \right|_{\text{surface}} = \frac{k_m L}{D_e} \theta|_{\text{surface}} = N_{Bi,m} \theta|_{\text{surface}}$$

In this case, we should take a practical look at the terms in the Biot number and see if it is stuck in a particular range (always large, always small, etc.).

$$N_{Bi,m} = \frac{k_m L}{D_e}$$

$$k_m = \frac{D_{fluid}}{\delta_m} \text{ (stationary interface with no interface reaction)}$$

$$N_{Bi,m} = \frac{D_{fluid}}{D_e} \frac{L}{\delta_m}$$

The boundary layer is the largest it can be in a stagnant system, and in that case it is approximately the same length scale as L . (Consider the example of the Sherwood or Nusselt number for spheres in a stagnant medium.) Thus the ratio L/δ_m is typically greater than one.

Also, the diffusivity of the species in the fluid is typically much greater than the effective diffusivity in the particle, where the species has to navigate the tortuous pore space. Thus D_{fluid}/D_e is also typically much greater than one.

Let's consider the mass transfer Biot number definition again and consider what it means in words:

$$N_{Bi,m} = \frac{D_{fluid}}{D_e} \frac{L}{\delta_m} = \frac{L/D_e}{\delta_m/D_{fluid}} = \frac{\text{"internal diffusion resistance"}}{\text{"external diffusion resistance"}} \gg 1$$

So the consequence of this ratio is that whenever we are looking at a reactor where we have bulk kinetic data, if we find out that we have an external diffusion limitation, we know that we must also have an internal diffusion limitation.

However, we might not be all that concerned, because when you have an external limitation, you may be able to predict the reaction rate purely by considering the surface flux:

$$-r'' = W = k_m (c_b - c_s) \approx k_m c_b \text{ because } c_s \ll c_b \text{ when external limitations prevail.}$$