

# Hamiltonian dynamics and neural networks

# Harmonic oscillator

$$\begin{aligned}\dot{p} &= -x - D\dot{x} \\ \dot{x} &= p\end{aligned}$$

spring force

friction

position  $\longleftrightarrow$  inhibitory neuron  
momentum  $\longleftrightarrow$  excitatory neuron

# Dissipation

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial x} - D\dot{x}$$

$$\dot{H} = \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial x} \dot{x}$$

$$= -D\dot{x}^2$$

# Symmetric network

- Equivalent if  $p=b+Wx$

$$\dot{x} + x = f(b + Wx)$$

$$\dot{x} + x = f(p)$$

$$\dot{p} + p = b + Wf(b + Wx)$$

$p = b + Wx$  is an invariant manifold

$$\begin{aligned} \left(1 + \frac{d}{dt}\right)(b + Wx - p) &= b + W(x + \dot{x}) - (p + \dot{p}) \\ &= b + Wf(p) - b - Wf(b + Wx) \\ &= W[f(p) - f(b + Wx)] \end{aligned}$$

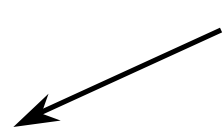
# Hamiltonian form

$$H = \mathbf{1}^T F(p) - p^T x + \mathbf{1}^T \bar{F}(x) \\ - b^T x - \mathbf{1}^T F(b + Wx) + \mathbf{1}^T \bar{F}(x)$$

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial x} - 2[f^{-1}(x + \dot{x}) - f^{-1}(x)]$$

friction



# Energy dissipation

$$\begin{aligned}\dot{H} &= \dot{x}^T \frac{\partial H}{\partial x} + \dot{p}^T \frac{\partial H}{\partial p} \\ &= -\dot{x}^T \left\{ \dot{p} - 2 \left[ f^{-1}(x + \dot{x}) - f^{-1}(x) \right] \right\} + \dot{p}^T \dot{x} \\ &= -2\dot{x}^T \left[ f^{-1}(x + \dot{x}) - f^{-1}(x) \right] \leq 0\end{aligned}$$

# Antisymmetric networks

- Equivalent if  $p=b+Ax$

$$\dot{x} = f(b + Ax)$$

$$\dot{x} = f(p)$$

$$\dot{p} = Af(b + Ax)$$



$p=b+Ax$  is an invariant manifold

$$\begin{aligned}\frac{d}{dt}(b + Ax - p) &= A\dot{x} - \dot{p} \\ &= A[f(p) - f(b + Ax)]\end{aligned}$$

# Hamiltonian form

$$H = \mathbf{1}^T F(p) + \mathbf{1}^T F(b + Ax)$$

$$\frac{\partial H}{\partial p} = f(p)$$

$$\begin{aligned} -\frac{\partial H}{\partial x} &= -A^T f(b + Ax) \\ &= Af(b + Ax) \end{aligned}$$

# Excitatory-inhibitory networks

conjugate variables

$$\tau_x \dot{x} + x = f(u + Ax - By)$$

$$\tau_y \dot{y} + y = g(v + B^T x - Cy)$$

conjugate variables

# Phase space dynamics

phase space  
( $x, y, p_x, p_y$ )

$$\tau_x \dot{x} + x = f(p_x) \quad \left( r + \frac{d}{dt} \right) (u + Ax - By - p_x) = 0$$

$$\tau_y \dot{y} + y = g(p_y) \quad \left( r + \frac{d}{dt} \right) (v + B^T x - Cy - p_y) = 0$$



attractive  
invariant  
manifold

$$p_x = u + Ax - By$$

$$p_y = v + B^T x - Cy$$

state space  
( $x, y$ )

$$\tau_x \dot{x} + x = f(u + Ax - By)$$

$$\tau_y \dot{y} + y = g(v + B^T x - Cy)$$

# Hamiltonian form

$$H = \tau_x^{-1} \Phi(p_x, x) + \tau_y^{-1} \Gamma(p_y, y) + rS(x, y)$$

$$\dot{x} = \frac{\partial H}{\partial p_x}$$

$$\dot{y} = \frac{\partial H}{\partial p_y}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} + A\dot{x} - B\dot{y} - (\tau_x^{-1} + r) \left[ f^{-1}(\tau_x \dot{x} + x) - f^{-1}(x) \right]$$

$$\begin{aligned} \dot{p}_y = & -\frac{\partial H}{\partial y} + B^T \dot{x} - C\dot{y} - (\tau_y^{-1} - r) \left[ g^{-1}(\tau_y \dot{y} + y) - g^{-1}(y) \right] \\ & + 2r\dot{y}^T (v + B^T x - Cy - p_y) \end{aligned}$$