MIT Department of Brain and Cognitive Sciences
9.641J, Spring 2005 - Introduction to Neural Networks
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Hamiltonian dynamics and neural networks

Harmonic oscillator

spring force

$$\dot{p} = -x - D\dot{x}$$

$$\dot{x} = p$$

friction

position $\longleftrightarrow$ inhibitory neuron

momentum $\longleftrightarrow$ excitatory neuron

Dissipation

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial x} - D\dot{x}$$

$$\dot{H} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial x}\dot{x}$$

$$= -D\dot{x}^2$$

Symmetric network

- Equivalent if *p*=*b*+*Wx*

$$\dot{x} + x = f\left(b + Wx\right)$$

$$\dot{x} + x = f\left(p\right)$$

$$\dot{p} + p = b + Wf\left(b + Wx\right)$$

$p=b+Wx$ is an invariant manifold

$$\left(1+\frac{d}{dt}\right)(b+Wx-p) = b+W(x+\dot{x})-(p+\dot{p})$$

$$= b+Wf(p)-b-Wf(b+Wx)$$

$$= W\left[f(p)-f(b+Wx)\right]$$

Hamiltonian form

$$H = \mathbf{1}^T F(p) - p^T x + \mathbf{1}^T \overline{F}(x)$$
$$- b^T x - \mathbf{1}^T F(b + Wx) + \mathbf{1}^T \overline{F}(x)$$

$$\dot{x} = \frac{\partial H}{\partial p}$$

friction

$$\dot{p} = -\frac{\partial H}{\partial x} - 2\left[f^{-1}(x + \dot{x}) - f^{-1}(x)\right]$$

Energy dissipation

$$
\begin{aligned}
\dot{H} &= \dot{x}^T \frac{\partial H}{\partial x} + \dot{p}^T \frac{\partial H}{\partial p} \\
&= -\dot{x}^T \left\{ \dot{p} - 2\left[f^{-1}(x+\dot{x}) - f^{-1}(x) \right] \right\} + \dot{p}^T \dot{x} \\
&= -2\dot{x}^T \left[f^{-1}(x+\dot{x}) - f^{-1}(x) \right] \le 0
\end{aligned}
$$

Antisymmetric networks

- Equivalent if *p*=*b*+*Ax*

$$\dot{x} = f\left(b + Ax\right)$$

$$\dot{x} = f\left(p\right)$$

$$\dot{p} = Af\left(b + Ax\right)$$

$p=b+Ax$ is an invariant manifold

$$\frac{d}{dt}\left(b+Ax-p\right)=A\dot{x}-\dot{p}$$

$$=A\left[f\left(p\right)-f\left(b+Ax\right)\right]$$

Hamiltonian form

$$H = \mathbf{1}^T F(p) + \mathbf{1}^T F(b + Ax)$$

$$\frac{\partial H}{\partial p} = f(p)$$

$$-\frac{\partial H}{\partial x} = -A^T f(b + Ax)$$

$$= Af(b + Ax)$$

Excitatory-inhibitory networks

conjugate variables

$$\tau_x \dot{x} + x = f\big(\overbrace{u + Ax - By}\big)$$

$$\tau_y \dot{y} + y = g\big(\underbrace{v + B^T x - Cy}\big)$$

conjugate variables

Phase space dynamics

phase space
(x, y, p_x, p_y)

$$\tau_x \dot{x} + x = f(p_x) \quad \left(r + \frac{d}{dt}\right)(u + Ax - By - p_x) = 0$$

$$\tau_y \dot{y} + y = g(p_y) \quad \left(r + \frac{d}{dt}\right)(v + B^T x - Cy - p_y) = 0$$

attractive
invariant
manifold

$$p_x = u + Ax - By$$

$$p_y = v + B^T x - Cy$$

state space
(x, y)

$$\tau_x \dot{x} + x = f(u + Ax - By)$$

$$\tau_y \dot{y} + y = g(v + B^T x - Cy)$$

Hamiltonian form

$$H = \tau_x^{-1}\Phi\left(p_x, x\right) + \tau_y^{-1}\Gamma\left(p_y, y\right) + rS\left(x, y\right)$$

$$\dot{x} = \frac{\partial H}{\partial p_x}$$

$$\dot{y} = \frac{\partial H}{\partial p_y}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} + A\dot{x} - B\dot{y} - \left(\tau_x^{-1} + r\right)\left[f^{-1}\left(\tau_x\dot{x} + x\right) - f^{-1}\left(x\right)\right]$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} + B^T\dot{x} - C\dot{y} - \left(\tau_y^{-1} - r\right)\left[g^{-1}\left(\tau_y\dot{y} + y\right) - g^{-1}\left(y\right)\right]$$

$$+2r\dot{y}^T\left(v + B^T x - Cy - p_y\right)$$