WHEN IS REASONING VISUAL?

GEORGE STINY
MIT

Abstract. How to stop counting and start seeing.

My line of thought

I'd like to take a stab at the following question from the point of view of someone who thinks about design. I'm mostly interested in how I can come up with something new when I calculate with shapes. How creative can I possibly be if I use rules? Am I any more creative if I don't? I won't say anything else about this, but keep it in mind. My question right now has to do with seeing:

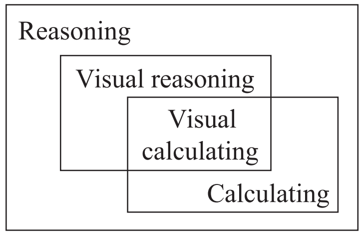
(1) When is reasoning visual?

That I come from MIT where the motto is 'Minds and Hands' — I'm told this means theory and practice — may put me at something of a disadvantage when it comes to original thinking about minds and eyes. Anyone who has seen MIT knows instantly that no one there pays much attention to how things look, even with a distinguished architectural tradition including Aalto and today Siza, Maki, Gehry, and Correa. MIT is not a visual place. I'm going to try and see anyway.

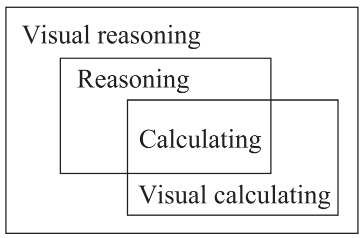
I'm not daring enough to attack question (1) directly. My first thought was to tackle two additional questions and to take a route through their answers that appeared to me more clearly marked:

(2) When is calculating visual?
(3) Does reasoning include calculating?

If the answer to question (3) is a solid yes and I can answer question (2), then I can use visual calculating as a working model to understand visual reasoning. This is reasoning by inference — from the properties of a part to the properties of the whole. And this is how everything is related



if it all fits together the way it's supposed to. But the longer I looked at my neat little diagram — is this visual reasoning? — the more I thought about which way the inclusions should go. Maybe the relationships are really like this



so that reasoning and calculating are merely special cases of visual reasoning and visual calculating. In fact, I'm sure each of my diagrams is correct in its own way, and have already established the underlying equivalencies for calculating by counting and alternatively by seeing. (Counting is the standard model. I'll return to the idea that calculating is normative later on.) Some of you may be familiar with the details. They involve the algebras of shapes $U_{i\,j}$

$$\begin{array}{cccc} U_{0\,0} & U_{0\,1} & U_{0\,2} & U_{0\,3} \\ & U_{1\,1} & U_{1\,2} & U_{1\,3} \\ & & U_{2\,2} & U_{2\,3} \\ & & & U_{3\,3} \end{array}$$

and whether basic elements in shapes are points with dimension $i = 0$, or lines, planes, or solids with dimension $i > 0$. And then there are a number of technical devices including, for example, analytic descriptions of basic elements and their boundaries, canonical representations of shapes using maximal elements, and reduction rules to compare shapes and combine them. But right now, I'd like to take a more leisurely, conversational approach. There's no reason to insist on rigor in Bellagio on Lake Como in the summer.

First I show that calculating is part of reasoning, and then by analogy — how many kinds of reasoning are there anyway? — conclude that visual calculating is part of visual reasoning. And I go on to present some of the evidence I've found for visual calculating. My plan is the same. What I glean from visual calculating will tell me what I need to know about visual reasoning. This is a discursive process that runs by logic or desultory rambling. I'm not too fond of logic and avoid it if I can. So I'm apt to ramble aimlessly. Roaming around to see what's what is a more effective way (procedure) to get new and useful results. But the ambiguity of this process — sometimes logic and sometimes not — isn't wasted. It shows better than anything I can say why visual calculating and so reasoning are important. They're the only way I know to deal with ambiguity and novelty, and not give up on calculating as a creative part of thought. This is the key if calculating is to model visual reasoning.

Some of you may have already decided that my line of thought is only engineering. It may be practical to use models and the like, but it's unlikely to lead to fresh insights of the kind you have come to hear. I won't argue that I'm not doing engineering. I'm more confident about calculating than I ever will be about reasoning. I can point to examples of calculating — including one or two of my own invention — but I'm never sure about reasoning. My own reasoning when it goes beyond calculating is as suspect as any. If I think I've got a really good argument, someone soon comes along and pokes holes in it. And it's the same if I try to follow the reasoning of others. I go from thinking I'm thinking to thinking I'm not. This is the sort of abstruse game philosophers like to play. It's hard, and it goes on and on forever. I'm no philosopher. I'm a lot happier with the more accessible pleasures of engineering. I like to calculate and to get sensible results.

But what about fresh insights? Everybody is always on the lookout for something new. Is there any reason to buy into my three questions if it's not going to go anywhere new and different? I'm always surprised at how much more there is to calculating than I expect at the beginning. What surprises me the most is that the best surprises don't come from clever ways of counting or complicated coding tricks that take real brain power — from what's valued and encouraged in calculating — but straight from seeing. Let's see just how visual calculating works. I can't be sure before I show you, but I'm almost positive that you're going to be surprised at how much there is to visual reasoning if it's like visual calculating.

Does reasoning include calculating?

I want to show that reasoning includes calculating. Most of the people I've asked agree that it does, but only as a narrow kind of process among many other kinds of greater scope that contribute more to thinking. Even so, I need to show this to go on: I need an account of reasoning that's like calculating, so that I can explain visual reasoning in terms of visual calculating. I won't develop an account of my own from scratch. Others have done this — Thomas Hobbes was apparently the first one — in a variety of different ways from which I can select. This may take some reasoning — at least a little judgment if not actual calculating — but the stakes aren't high. There's enough agreement to decide on aesthetic grounds alone. There's no reason for careful analysis and rigorous argument. This is the kind of everyday choice that's easy to make, but that's hard to justify beyond its results. This is the kind of choice I like.

The American pragmatist William James gives an account of reasoning in *The Principles of Psychology* that lets me start. For James, reasoning is a compound process with interlocking parts. He divides the 'art of the reasoner' into moieties:

> First, *sagacity*, or the ability to discover what part, M, lies embedded in the whole S which is before him;
> Second, *learning*, or the ability to recall promptly M's consequences, concomitants, or implications.

This is exactly what happens when rules are used to calculate. There's the part M to be embedded in the whole S and the consequences, etc. — call them P — of finding M in S. The rule M → P applies to S to produce something new. Of course, this is only a gloss. It leaves out most of the important details. I have to say a lot more about how the rule works when it's used. The real trick is to find a suitable embedding relation, and to show how M can be embedded in S, and how together with P, this changes S.

James takes the syllogism as his example — this only confirms the link to calculating — with almost no attention to the underlying details that make rules work. But he has something else far more interesting and weighty in mind. He wants to plumb creative thinking and describe the source of originality. In fact, James's overarching definition of reasoning is the ability to deal with novelty. Isn't this why reasoning — especially the visual kind — makes a difference?

> If we glance at the ordinary syllogism —
>
> M is P;
> S is M;
> ∴ S is P
>
> — we see that the second or minor premise, the 'subsumption' as it is sometimes called, is the one requiring the sagacity; the first or major the one requiring the fertility, or fulness of learning. Usually, the learning is more apt to be ready than the sagacity, the ability to seize fresh aspects in concrete things being rarer than the ability to learn old rules; so that, in most actual cases of reasoning, the minor premise, or the way of conceiving the subject, is the one that makes the novel step in thought. This is, to be sure, not always the case; for the fact that M carries P with it may also be unfamiliar and now formulated for the first time.

I think James is right about 'the novel step in thought'. Embedding — 'the ability to seize fresh aspects in concrete things' — is the key. And I'll give a more faceted account of the embedding relation as I go on. Of course, reasoning that works isn't always surprising, but it may be if it doesn't. Let's look at an example of this that shows a few details of embedding, and how rules make use of them. My example deals with line drawings (shapes), but this doesn't mean it's visual. Whether or not seeing relies on reasoning, calculating needn't involve seeing. In fact, my example shows how calculating and seeing may disagree. There's got to be a way to reconcile them if I'm going to show how visual calculating and hence reasoning are possible. Again, embedding is the key.

A first look at calculating

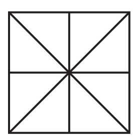
T. G. Evans — an aboriginal computer scientist — uses the rules of a 'grammar' to define shapes in terms of their 'lowest level constituents' — or alternatively atoms, components, primitives, units, and the like. This illustrates some notions that have been used widely in computer applications for a long time. But in fact, these ideas are as fresh today as they ever were. (Rules like this were applied early on in 'picture languages' to combine picture atoms and fragments, and later in Christopher Alexander's much better known but formally derivative 'pattern language'. My own set grammars are also comparable, even if more ambitious: they're the same as Turing machines.)

Evans's grammar contains rules like this one

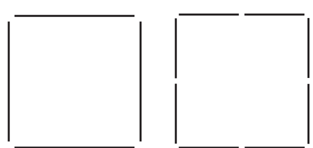
Three lines → Triangle

that defines triangles in the ordinary way as polygons with three sides. In order to show how the rule works, I first have to give the embedding relation, and then tell what options there are to satisfy it. For Evans, embedding is identity among constituents. (This is what happens when i is zero in the algebras U_{ij}.) The rule applies to a shape if all of it's constituents — or more generally for any rule in a recursive scheme, the constituents the rule implies — are also in the shape. The constituents in the rule may be transformed as an arrangement — moved around freely, reflected, or scaled — to obtain a correspondence with constituents in the shape.

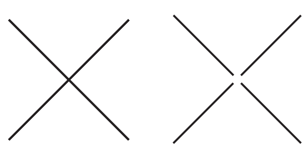
Evans uses this shape



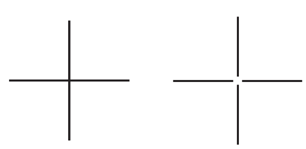
as an example. The shape has a finite set of constituents. It's 24 line segments that are each defined by its endpoints: first there are the four sides of the large square and their halves



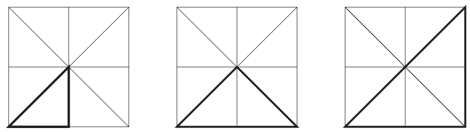
then the two diagonals of the square and their halves



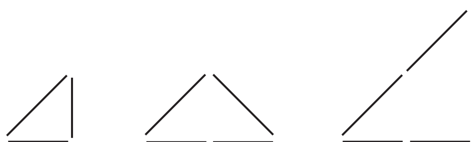
and finally the horizontal and the vertical and their halves



And Evans's rule — the one I showed above — applies without a hitch to pick out the 16 triangular parts of the shape: eight small triangles, four medium ones, and four large ones

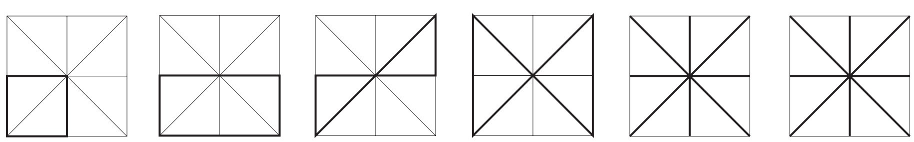


This is really pretty neat. In fact, it's very clever coding. If there were only eight long lines in the shape, then there would just be large triangles and no small or medium ones. And if lines were only halves, then there would be three distinct kinds of triangles



not all with three sides, that I would have to define in separate rules. This isn't what I know about triangles or what I actually see. But Evans's grammar gets around all of this artfully. The grammar is seeing what I do, when a solitary rule of no great complexity — merely three lines — is used to calculate. What else do rules and constituents imply? How far can I go with this?

I can give rules for squares and rectangles — four lines apiece — bow ties of distinct shapes — again four lines apiece — and visually homonymous stars — one type defined by four lines and the other type by eight halves

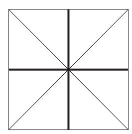


And my rules find all of these figures wherever they are in the shape: there are five squares, four rectangles, six bow ties, and a star of each type. This is great. But already, there are signs of trouble. I can't tell the stars apart just by looking at them, even if I can by the rule I apply. Calculating and seeing are beginning to look different.

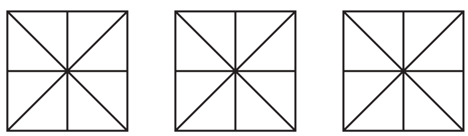
Now suppose I go on to define an additional rule

Line →

that erases lines — the lowest level constituents — in Evans's shape. What do I get if I apply my new rule to erase the lines in all small triangles, or medium or large ones? Whatever happens, I don't expect to see the lines I've removed in my result. If I do the erasing by hand to remove the lines I see in small triangles, the shape disappears. And for medium or large triangles — erasing what I see by hand — I get a Greek cross that looks like this

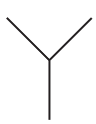


But in all three cases when I calculate with the rule, my results look like this



Seeing is believing. Some parts are hard to delete when I use my eyes, even after I've applied the rule. The shape is visually intact whichever lines I erase. You can check me if you like. I haven't made any careless mistakes, yet I don't get what I expect. I must be seeing things. Actually, I am. The lines I've erased with the rule aren't there. No two of the resulting shapes are the same — they're numerically distinct with eight, 16, and 18 constituents apiece — and they're all different than the shape I started with. But how can I tell which shape is which if I merely look? How can anyone, without consulting someone else? God might know, but I'm left with experts and quacks. Who do I ask to tell them apart? I'm lucky I don't have to trust my eyes. There's a better way. I can calculate to find out. I can keep track of where I've used the rule to distinguish what I can't see. This may not be the kind of novelty James has in mind.

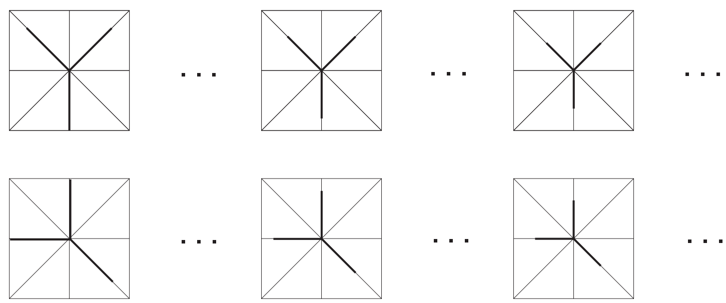
Seeing is confused in other ways as well, when it comes to calculating in Evans's grammar. Many parts that are easy for me to see — there are no tricks, you can see the same parts yourselves — are impossible for rules to find. Take the triaxial motif



with equilateral arms. I can define the Y — there are also other letters from A to Z — in the rule

Three lines $\rightarrow$ Y

The Y appears indefinitely many times

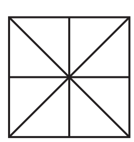


yet the rule — or any other rule — can't find it, not even once. All of my results are correct. This is calculating, but it isn't seeing. What's gone wrong? All of a sudden, Evans's grammar is blind. Is there always something that's going to be missed when I calculate? It's not because I can't define precisely what I want to find. I can do that for any shape I like. But surely, there's got to be some way to see Y. Maybe it should be defined in the rule

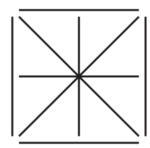
Three lines $\rightarrow$ WHY?

This takes me to real homonyms. What I thought at first was seeing isn't seeing at all. It's hearing. Sensible experience is much the same from one modality to the next. It's ambiguous. Parts are forever fusing and dividing. Is this going too far? I don't know. I don't think James would think so. It's more than likely that he would welcome the novelty. One of his students gave us the notorious line 'A rose is . . . a rose'. Why can't I see Y when I calculate?

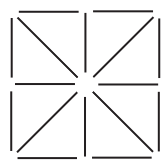
Let's see how the shape



is put together, and what this implies about what I can see. The 24 constituents (line segments) in the shape include eight long lines

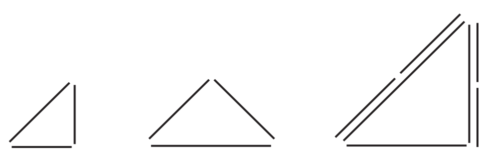


and their 16 halves



Each long line contains two halves, and each half is contained in a long line. This gives me the 16 triangles I want. But if I erase a long line, there are two halves that visually compensate for the loss. And if I erase any half, there's a long line that fills in. The shape is going to look exactly the same, even as its constituents change. Calculating multiplies distinctions I simply can't see.

(A productive way to explore the depths of Evans's shape is to count the different kinds of triangles it contains. Small triangles always have three lines, medium ones have three to five lines, and large ones have three to nine lines. Lines are taken from these schemes



There are five distinct configurations for each grouping of three collinear lines

So there are five versions of each medium triangle, and 125 of each large one. The census is given in Table 1. More is happening in Evans's shape than I can possibly see. What use are my eyes when nearly everything is hidden? I've no doubt that Evans's shape contains 16 triangles with three lines apiece. Seeing is believing. But it's only a half — actually 3% — truth.)

Table 1

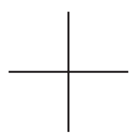
Census of Triangles in Evans's Shape

Number of Lines	3	4	5	6	7	8	9	3-9
Number of Triangles	16	48	124	180	120	36	4	528

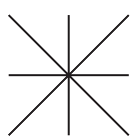
Of course when I erase a long line, I can also remove it's halves at the same time. This takes another rule

Three collinear lines →

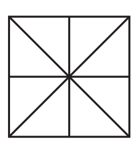
but gives some interesting results. I get a Greek cross (two lines) and another one (four halves) — at least this looks right — erasing the sides of large triangles



a star (four lines) and a Greek cross (four halves) — or maybe the star is a pair of crosses (two lines apiece), but that's OK — for the sides of medium triangles



and this — it's eight lines — for the sides of small triangles



This is a marked improvement, and it suggests more. If I add the conjugate rule for halves

Two collinear lines →

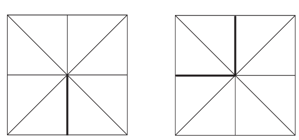
so that the long line that contains a half is erased at the same time the half is, then I get the visual results I'm looking for. I started with a single rule to erase a line, and now have three rules that have to be applied judiciously. But so what? The grammar is fixed, even if the new rules are ad hoc.

(This reminds me of the old joke about an MIT engineer. In fact, Evans hails from MIT, too. Anyway, the engineer is going to be guillotined. The blade falls fast and stops scarcely short of his neck. He looks up and points — 'I can fix that!' And so he does to show he's sharp and to collect his reward. Poor sap: nobody told him that hard problems — the only kind worth thinking about at MIT — don't necessarily have useful answers. No matter what kind of research you do, it's got to be very complicated. This shouts, 'Hey! — I'm from MIT.' Good ideas aren't meant to be interesting or fun. They have to be hard to show your worth. The more I think about visual calculating and how to do it, the more I'm sure that it's the uncomplicated, the vague, and the ambiguous that matter the most in both research and education. It's much better to be flexible than tough. Shapes are full of miscellaneous possibilities. You never know what else there is to see.)

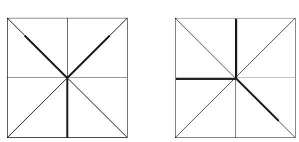
But what about the Y? Why can't the rule

Three lines $\rightarrow$ Y

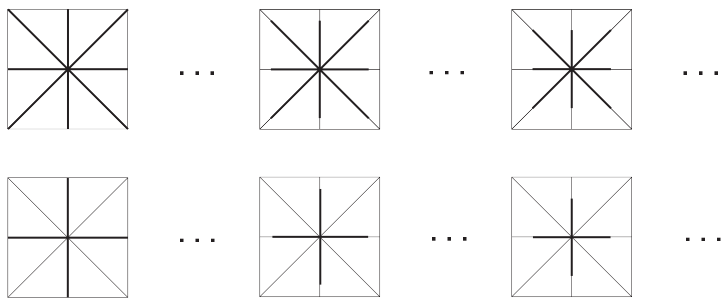
find it when I can see it almost everywhere I look? While there may be too many constituents to erase the lines in triangles, there aren't enough constituents to find even a single Y. One or two segments can be found in a number of places



but never three segments at once. New constituents can always be added to complete Y's



(What do additional constituents do when I try to erase the lines in triangles? How do they change the census of triangles in Table 1?) Yet no matter what I do, there are never going to be enough constituents to find all of the Y's there are to see. I can't specify (anticipate) everything I might see before I've had a chance to look. What about other letters from A to Z? What about big K's and little k's? What about stars and crosses

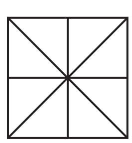


Evans's constituents miss most of them. My analysis has got to stop sometime, and when it does, calculating goes blind. Analysis may be an essential part of reasoning — maybe it's a prerequisite — but it seems only to get in the way of seeing.

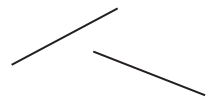
Neither the surplus of lines nor the lack of Y's — and other parts — seems right. What I see and what Evans implies I should when I calculate are just not the same. Must calculating and visual experience be related so haphazardly? There's a huge gap to bridge between minds and eyes if I'm ever going to get anywhere with visual calculating. Visual reasoning is long out of sight. But hold on. Is the gap real, or an example of not thinking that can be addressed without appealing endlessly to ad hoc devices?

The problem with Evans's grammar is completely artificial — if you're still with me, you've probably decided all calculating is — and it needn't really occur. Evans uses a zero dimensional embedding relation that's meant for points to calculate with shapes made up of one dimensional line segments. (This amounts to confusing the algebras U_{ij} for i = 0 and i = 1.) This appears trivial enough, but the disparity has two bewildering consequences.

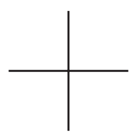
First, the shape



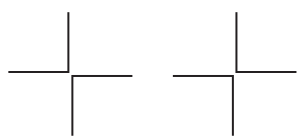
has no obvious constituents to serve as points. It's easy to see that lines come undivided. How can I cut these two



in a meaningful way without knowing beforehand what rules there are and how I'm going to use them? Internally, the lines are homogeneous. And externally, their relationship is arbitrary. This is how it is for lines in any arrangement. I just can't be sure how many. Lines aren't numerically distinct like points. They fuse and divide freely. I can draw two lines to make a Greek cross

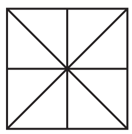


and then see alternating pairs of L's



or I can turn this around and draw L's — two or more — and see a Greek cross. Nevertheless, I've decided to calculate. So I've got to define points explicitly in whatever way I can think up according to my present interests and goals. Right now, I'm looking for triangles. Is this circular reasoning or what? Reasoning obviously comes in shapes of various kinds. If I look the right way, this circle appears almost hermeneutic. I've got to find triangles to define a rule to find triangles, or something along these lines. Seeing and calculating are linked. That's for sure. But perhaps their relationship is not what I want. Seeing may be finished — completed if not done for — before I calculate at all. What would calculating be like if there were no problem to solve before calculating to solve a problem?

And second, after I determine these points, I have to keep to them for as long as I calculate. There's no way to start over with another analysis. I'm told constantly that it's cheating if I do. It's incoherent. (Emerson is famous for the opposite view. 'A foolish consistency is the hobgoblin of little minds . . . With consistency a great soul has simply nothing to do.' Today the first sentence is only a cliché. As a student I used it as a reason to be silly. But the following sentence is the real clincher. If the analysis I've already got is a special description that controls my ongoing experience, then what am I free to do? There are no surprises left. Why not change my mind? What stops me from seeing something new? I can't imagine a consistency that's not foolish sooner or later if I calculate with shapes. What I see may alter erratically.) There's an underlying description of the shape



that depends on how I've defined constituents before I begin to calculate. The description isn't anywhere to see. It's hidden out of sight and isn't supposed to intrude. Only it limits everything I do and everything I see. There's no problem, so long as I continue to experience the same things and my goals stay the same — if nothing ever changes. Otherwise, it's nearly impossible to try anything new — to erase lines or find Y's — and not run into trouble.

Evans isn't entirely to blame for this choice of embedding relation. He uses concepts like 'three lines are a triangle' to describe what he sees in the expected way when it comes to making computer models. This amounts to the following: take things that aren't zero dimensional and go calculate with them as if they were. There are abundant examples of this — from computer graphics, imaging, and fractal modeling, to engineering analysis and weather forecasting, to complex adaptive systems of every kind. Finite elements — atoms, components, lowest level constituents, primitives, units, etc. — are combined to describe things that aren't sensibly divided. It's a very powerful method with an ancient history and an amazing record of success. It's what most thinking is like at MIT, and it works like magic. That's right, it's an illusion. It may be exposed when calculating and sensible experience are asked to agree too closely. Only this isn't new. James has his own examples of the 'many ways in which the conceptual transformation of perceptual experience makes it less comprehensible than ever.' Have I taken a wrong turn? Maybe there's no such thing as visual calculating. Are clear negative examples reason to give up, or reason to try another way? What kind of evidence would count otherwise? I have to show that analysis isn't something you do first in order to calculate, but rather something that changes or evolves — even discontinuously — as one of the by-products of calculating. Analysis develops as calculating goes on. What I do while I calculate and not before — what rules I've got and what I do with them — determines what constituents are.

When is calculating visual?

Sometimes I try an informal rule of thumb to decide when calculating is visual. Both the dimension (dim) of the elements (el) and the dimension of the embedding relation (em) used to calculate are the same. I can state this in a nifty little formula that's a good mnemonic:

dim(el) = dim(em)

I'm pretty sure the rule is sufficient: whenever my formula is satisfied, it's visual calculating. I'm just not so sure the rule is necessary. There may be examples of visual calculating that don't meet this standard. Whatever the answer — and I'm ready to bet on the formula — I want the equivalence (biconditional)

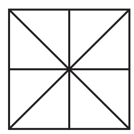
dim(el) = 0 ≡ dim(em) = 0

to be satisfied, to ensure that zero dimensional embedding relations are only used with zero dimensional elements.

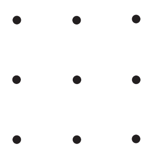
As formulas go, mine is pretty vague. I don't say how to evaluate either side except in a few ad hoc cases. But all of my examples of visual calculating are synthetic, so the formula is enough for the time being. There's no reason to avoid vague ideas when they stimulate calculating. In fact, vagueness may be indispensable to what I'm trying to show. I can't imagine anything vaguer — or more ambiguous — than a shape that isn't divided (analyzed) into constituents, so that it's without definite parts and any obvious purpose. I'm going to use my formula to get to the idea that calculating is visual if it can deal with shapes like this. I want to use rules to determine what parts I see and what I can say about them, and to allow for what I see and what I say to change freely as I calculate. And I want this to happen every time I try a rule. I'm told calculating is a good example of what it means to be discursive: sometimes when I calculate it looks logical, but most of the time it's only desultory rambling with my eyes. Shapes should be ambiguous and vague, and ready to use when I calculate.

So what can I do to make Evans's example visual? My formula provides twin options. I can change the elements in Evans's shape from lines to points, so that dim(el) = dim(em), or I can use another embedding relation that's one dimensional. Each of these alternatives is feasible and amply rewards further attention.

Suppose that the shape



is the nine points Evans uses to define line segments as constituents

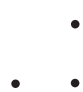


and that the embedding relation is unchanged: it continues to require identity among constituents. I really don't have too much choice. There's no other way to do embedding for points. (To be exact, I'm going to calculate with shapes in the algebra $U_{0\,2}$.)

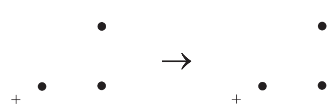
I can use the rule

Three points → Triangle

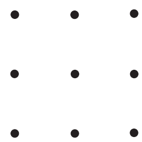
in place of Evans's rule to define 45° right triangles



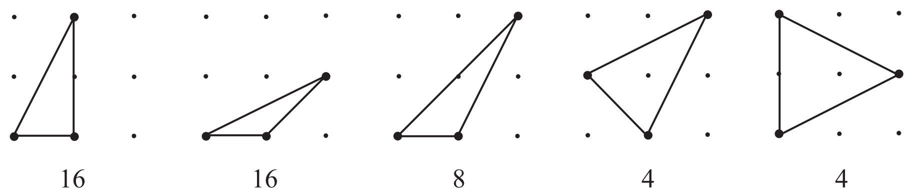
Or equivalently, I can give the rule in this identity



in the way I normally do, where two shapes — in this case, they're the same — are separated by an arrow. I'll say more about rules like this a little later on when I look at embedding for lines. My new rule in whatever form finds 28 different triangles in the shape



including the 16 from Evans's example. But there are 48 other triangles in the shape in five different constellations

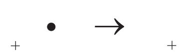


These are readily defined in additional rules. (Of course, it's easy to define all triangles using a single rule in the way Evans does, or as effortlessly, using a schema for rules in my way. This is also something I'll come back to again.) I can't find any other triangles. My grammar is seeing what I do.

Now what happens when I erase points or look for Y's? In the first case, my rule is just like it was before, but for points

Point $\rightarrow$

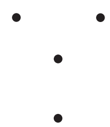
Or equivalently



When I apply the rule to erase the vertices of Evans's small triangles, the shape disappears the way it should. And if I do the same for medium triangles and for large ones, then I get the shapes

Everything looks fine, even if the Greek cross appears in alternative ways. In fact, this may be an advantage. I can decide whether I've erased the vertices of medium triangles or large ones simply by looking. There's nothing to see that I can't understand. Points aren't like lines. They don't fill in for the loss of others because they don't combine to make other points and don't contain them. The grammar I've got for points is doing a lot better than Evans's grammar for lines. Visual calculating may be a real possibility after all.

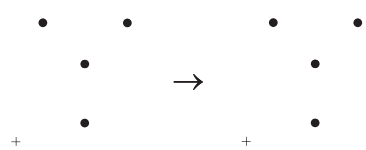
But what about Y's



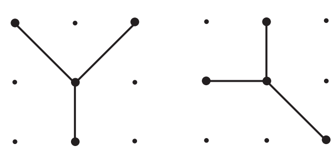
with equilateral arms that are defined in this rule

Four points → Y

or with the identity

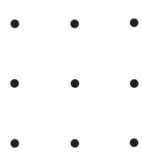


There's no problem. I can't see the Y anywhere in my shape. It's possible there are other kinds of Y's — I can define additional rules to find them — but their arms are different lengths



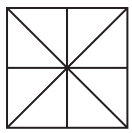
Calculating and seeing match again. This time with uncanny precision.

So here I am with the shape



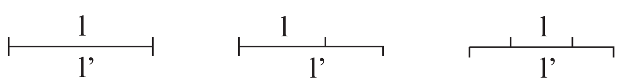
that's just what calculating tells me it is. Nothing is hidden. Anything a rule can find, I can see. And what I see depends on the rule I apply. There's nothing to see that a rule can't find. This is just what I had in mind for visual calculating. But my example isn't completely convincing. Everything works because the shape contains points — lowest level constituents again — that don’t interact. They’re independent in combination: embedding means identity. The whole thing is a strictly combinatorial affair — points are like marbles ready to count — and seeing is simply a matter of combining and rearranging them in alternative ways. Surely, there’s more to seeing than counting. To see what it is, I have to go back to lines and change the embedding relation somehow, so that it’s not merely identity.

Suppose I keep Evans’s shape the same



I can draw it with eight lines — its maximal elements — the way an experienced draftsman would — only four of Evans’s triangles are explicitly defined — but I’m not obliged to treat these lines as constituents. The parts of the shape may be combinations of maximal elements, or including everything I can, combinations of elements or their segments. Now there are indefinitely many parts, and none is preferred to any other. This is in fact the potential infinite of Aristotle — I’m free to divide the shape wherever I like — and not an actual infinite of parts that have been collected together. The parts I see — the limited number I resolve when I divide the shape — depend on the rules I’ve got and how I use them. I need to have an embedding relation for this that’s one dimensional and works for lines.

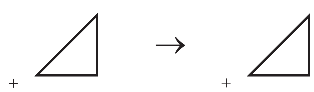
Let’s agree that a line l is embedded in any other line l’ if l and l’ are identical or — going beyond embedding for points — l is a segment of l’



More generally, a shape M is part of any shape S if the maximal elements of M are embedded in maximal elements of S. (I was drawn to this relation by necessity. When I started using a typewriter — I was eleven or so — I contrived an easy way to check my work against an original or if I had to retype a page to correct mistakes and not make more. Proofreading was hard. It took too long and was unreliable. So I’d embed one page in another. I’d tape the first to a window and move my copy over it. I could see what was on both pages at once. If things that were supposed to be the same lined up — at least piecewise — then I knew my typing was all right. Creative designers use yellow tracing paper in like ways whether or not they apply rules.) Whatever I can see in S —

anything I can trace — is one of its parts. This leads straight to calculating in the algebras of shapes $U_{i\,j}$, and is almost exactly what James has in mind when he describes how reasoning works. For the rule M → P and the shape S, if I can see M in S, then I can replace it with another shape P by subtracting M from S and adding P. I'll add by drawing shapes together, so that their maximal elements fuse. And I'll subtract by adding shapes and then erasing one, so that segments of maximal elements are removed. Remarkably, the embedding relation implies nearly everything I need in order to do all of this with axiomatic precision. I also need transformations of some kind to complete the correspondence between M and P, and S. But I'll skip the details because they're not hard to fill in, even if they're not without many important consequences. (Now I'm ready to calculate in the algebra $U_{1\,2}$ instead of the algebra $U_{0\,2}$. And I'll do everything in $U_{1\,2}$ from now on.)

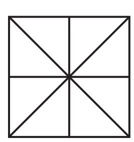
With lines and this embedding relation, Evans's rule to define triangles is simply the identity



I don't have to say what a triangle is in terms of constituents that are already given. I only have to draw it. It makes no sense to have the rule

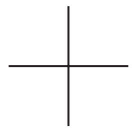
Three lines → Triangle

because I don't know how the sides of triangles are divided — remember, there's no telling how many lines there are in a shape — or even if sides are parts. And I don't have to divide Evans's shape into constituents either for the identity to apply in the way I want to find every triangle. The shape is OK as a drawing, too

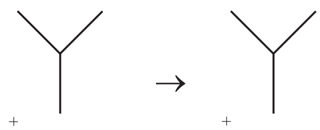


When I use the identity, there are 16 triangles — this is just what I see — even though the small triangles and the medium ones have maximal elements that aren't maximal elements in the shape. My erasing rule — the one I added to Evans's grammar — is

and it produces the results I get if I erase the lines I see by hand. If I apply the rule to the sides of small triangles, the shape disappears. And when I use it to take away the sides of medium or large triangles, I get the Greek cross



This is the way it's supposed to work — seeing and calculating agree perfectly. When I see a part and change it according to a rule, there's nothing hidden to confuse the result. Whatever distinctions a rule makes, I can see. There may be surprises — there are plenty to come — but they aren't artificial ones caused because I've misrepresented shapes with constituents that I can't redefine after I've started to calculate. There's no underlying analysis that determines what I can see and what I can do. Whatever is surprising is perceptual. It's a natural part of sensible experience. And the rule

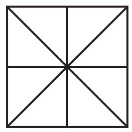


— another identity — finds as many Y's in the shape as there are to see. Seeing is never disappointed. There's no part I can see that a rule can't find. Novelty is always possible. What you see is what you get.

It may be useful to stop briefly to compare Evan's example for points and for lines. For points, there are finitely many parts for rules to find, but for lines, there are indefinitely many parts. This is evident for Y's, and fundamentally so for the rules

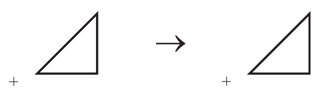
While the one can only be used a limited number of times, the other's work may never be done. A point — just like a constituent — can be erased in exactly one way. I can't remove some of it now and some of it later. It goes all at once. It's there or it's not. But I can divide a line wherever I like. I can take any piece now and another piece later. Its segments provide endless possibilities. Points and lines just aren't the same with their distinct dimensions and their corresponding embedding relations. If I confuse them in order to calculate, then I'm simply not calculating visually.

But there's a lot more. What about the constituents of the shape



How do these change as I calculate, and how are they finally defined? This is Evans's problem in reverse. He has to divide the shape before calculating, so that his rule can find all of the triangles there are to see. This analysis isn't part of calculating. It's more important. It's what makes calculating possible. I'd like to know how to define constituents, but Evans doesn't say. He probably doesn't think about it. It's just something you do when you're going to calculate. Even if it's obvious, I'd like to see what's involved. And it may not be obvious. Evans gives 24 constituents when 22 will do. So I'll take a different tack and show how to define constituents as part of calculating. I'm not stuck the way Evans is. I can look at everything — including analysis — as the result of applying rules. This is what I said visual calculating should do.

Let's start with something easy. I want to use the rule



to calculate, so that constituents are defined dynamically again and again in an ongoing process. The rule is an identity: it simply tells me a triangle is a triangle without referring to its sides — Evans's definition — or any other parts. I can't say anything about a shape in a rule — identity or not — that goes much further than pointing to it and announcing that it's that. I can name it — a triangle is a triangle — but there's nothing definite about its parts. I'll say more about this and what it means for visual calculating and reasoning itself as soon as I get to schemas for rules. Right now, though, I only have my identity for triangles. What good is it to know that a triangle is a triangle?

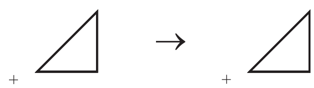
Identities are interesting rules. Rules are supposed to change things, but identities don't. Whenever they're used to calculate — let's apply one or more to a shape S a number of times — the result is a monotonous series that looks like this

$$S \Rightarrow S \Rightarrow \ldots \Rightarrow S$$

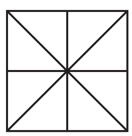
In each step ($S \Rightarrow S$), another part of S is resolved — that's the part I see —

and then nothing else happens. The shape doesn't change. It stays the same. At least that's how it looks. Identities are constructively worthless. They have no use. And it's standard practice to discard them. But this may be rash. It misses what they really do. There's far more to identities than idle repetition. Identities are observational devices. They're all I need to divide the shape S with respect to what I see. If I record the parts they pick out as they're tried, then I can define topologies for S that show its constituents and how they change as I calculate.

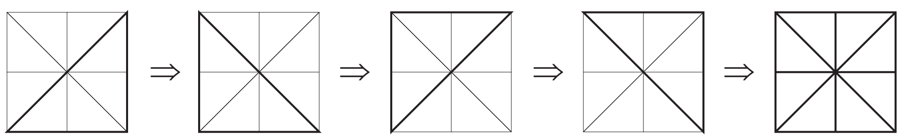
Suppose I apply the identity



to the shape

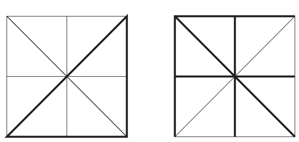


so that large triangles are picked out in a clockwise fashion in this four step series

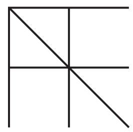


If I take the triangles the identity resolves — remember, these are the triangles I see — I can use them to define constituents. I'm going to calculate some more to explain how I've been calculating. There are a number of ways to do this. For example, I can work out sums and products, or I can add complements as well. Complements give Boolean algebras for the shape. They're a special kind of topology with atoms that provide a neat inventory of constituents.

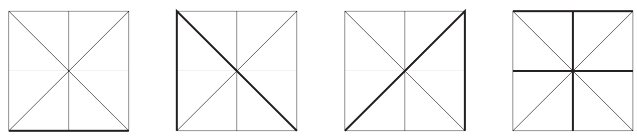
The first time I apply the identity to calculate, I get a Boolean algebra with two atoms



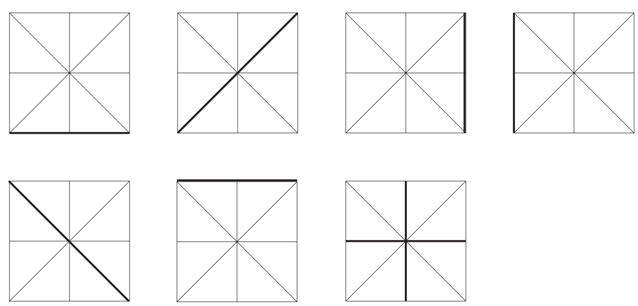
I see a triangle and it's complement. It may be a little farfetched to assume that I see the shape



— I have a hard time making sense of it — but complements are sometimes like this. They're surely worth the added effort, though, because they simplify things as I go on. There are four atoms when I apply the identity the second time

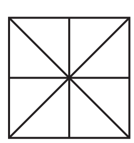


Now I see a pair of triangles and their complements that combine in products to define constituents. And there are seven atoms the third and fourth times I use the identity

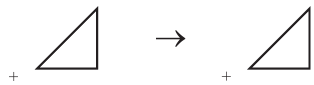


including the six individual sides of the large triangles and the interior Greek cross. This is all pretty good. The constituents I finally get match my visual expectations. The sides of the triangles interact, while the Greek cross is an independent figure.

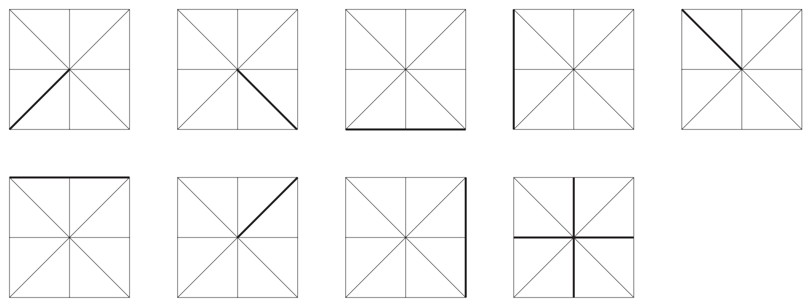
Of course, there are also medium and small triangles in the shape



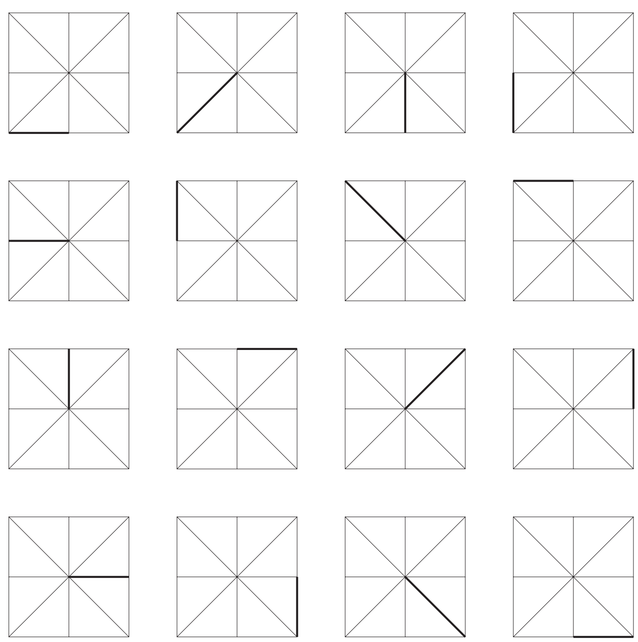
And I can apply the identity



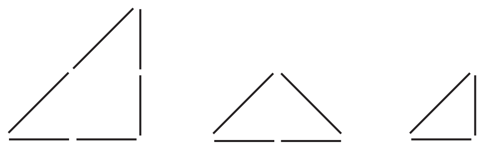
to find these triangles in exactly the same way. Medium triangles — if I resolve all of them — define nine constituents



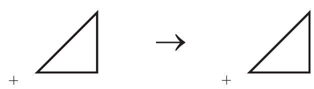
and small triangles — again, if I resolve all of them — define 16



The atoms I get for the small triangles — the half lines in Evans's example — are the most refined constituents I can define, no matter how the identity is used to pick out triangles. All of the large triangles in the shape contain six constituents, all of the medium triangles are made up of four, and all of the small triangles are made up of three



But this doesn't mean — as it did in Evans's example — that triangles have to be described (represented) in different ways for the identity to work as it's meant to. For the identity

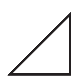


the embedding relation is given for lines, so that they can fuse and divide in any way whatsoever, but for Evans's rule

Three lines → Triangle

embedding requires that constituents — they neither fuse nor divide — match in an exact correspondence, as elements do in sets. Perhaps this is another way of seeing it. A shape — at least with lines — is not a set.

The identity applies to shapes, not to their descriptions. As far as the identity is concerned, a triangle is whatever I draw. It's simply this

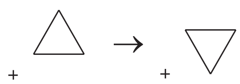


with no divisions of any kind: no sides, no angles, and no anything else. And it's always there to see if it can be embedded. Evans gets into trouble because of the way he calculates. He confuses a triangle — something sensible — with a solitary description — an abstract definition — that's only one of many. This is the kind of thing that can happen if my formula dim(el) = dim(em) isn't satisfied, and in particular, if the embedding relation is zero dimensional and elements aren't. Shapes and descriptions are different sorts of things. But more important, I can calculate with shapes without referring to their descriptions. This idea provides another way to think about visual calculating. But first, let's look at an example that shows a little more of what's at stake. It begins to fill in the picture for rules that aren't identities.

A second look at calculating

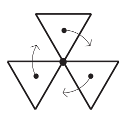
I'm positive that some of you have already seen the following example. Nevertheless, I'm going to show it again. Seeing something for a second or third time with altered emphasis may bring increased insight. After all, I'm dealing with shapes. And I want you to see how shapes are redescribed as I calculate. There's no given way to see shapes that rules can't change when they're tried. The ability to handle ongoing changes like this — there's always going to be a chance for more, even with only a single rule — is what makes calculating visual.

The rule



rotates an equilateral triangle about its center, so that this point is permanently fixed. I can apply the rule eight times to define the shapes in this series

It's easy to see that the final shape in the series is a rotation of the initial shape about its center. So far, so good. But If you think about it, this simply can't be right. The centers of the triangles in the initial shape change position as the initial shape is rotated



The rule doesn't move the centers of triangles, but they move anyhow. What's going on? How did this happen? It's more than a surprise. It's a new kind of paradox. (Often when I describe the series, I'm accused of faking the rotation. An MIT engineering professor thought so — his colleague lost his head fixing a translation device — and others have, too. What fun!)

The answer is something to see. The rule can be applied to the fourth shape in the series in alternative ways. In one way, the rule picks out the three triangles that correspond to the three triangles in the initial shape. But none of these triangles is resolved in the other way. Instead, the rule divides the fourth shape into two triangles — the large outside triangle and the small inside one — that have sides formed from sides of the three triangles that come from the ones in the initial shape. The rule rotates the two triangles in turn to get the fifth and sixth shapes. Now the rule can be applied in alternative ways to the sixth shape in the series. Either the rule resolves both of the triangles that correspond to the ones in the fourth shape, or the three triangles — one at each corner of the sixth shape — that have sides formed from segments of sides of the triangles in this pair. The rule rotates the three corner triangles one at a time to reach the final shape.

Twice in this process, the rule changes what I see in a surprising way. Whenever lines are combined, they fuse, so that all prior divisions disappear. Then new divisions are possible anywhere. I can always redefine triangles — and so redescribe the shape — according to how I apply the rule. The way the rule works makes this feasible as I'm calculating, without outside intervention. I'm embedding triangles to determine parts — tracing them out — not matching predefined constituents. The rule applies neither locally — as it appears to at first — nor globally — as it appears to in different ways later on — but anywhere there are triangles independent of how they were actually made. So long as lines fuse, there's no history I can use to tell parts apart. It's all visual. And it's all calculating. (It may be elucidating to compare this with the way 'emergent' properties are usually described in cellular automata and complex adaptive systems of other kinds. It's not done with the same rules given to calculate. When I calculate, there's no reason to appeal to anything more than rules.)

The nine shapes in the series

are all made up of triangles. Their numbers summarize the action of the rule as it's applied from step to step. Look at the following three series

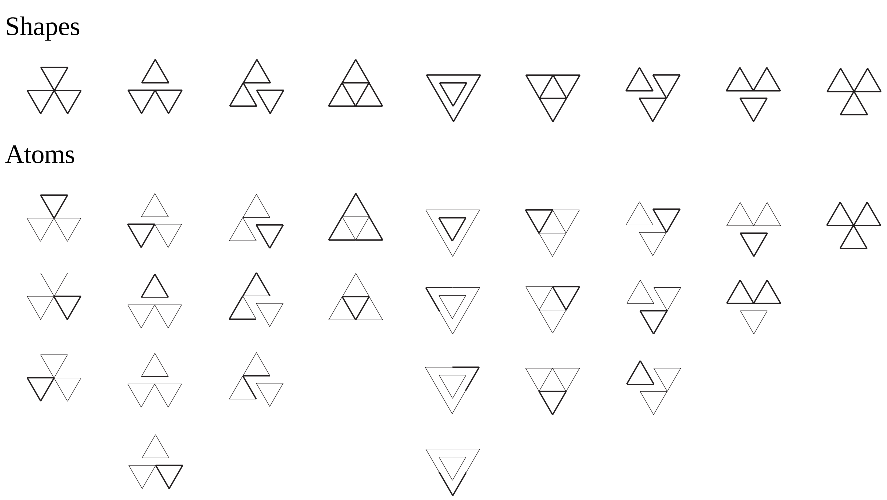
3	3	3	5	2	5	3	3	3
3	3	3	3	2	2	3	3	3
3	3	3	2	2	3	3	3	3

The first series shows the maximum number of triangles — these can be picked out using an identity — in each of the shapes. The next series gives the number of triangles in a shape after the rule has been applied, and the last series gives the number of triangles in a shape as the rule is being applied. In both of these cases, the number of triangles depends on what the rule does either by rotating an existing triangle or seeing a new one to rotate, and then counting in terms of the triangle and its complement. The resulting inconsistencies in the fourth and sixth shapes tell the story. There's no saying how many triangles there are until the rule is applied. There are triangles in both shapes — no one doubts it — but they're not numerically distinct. Counting goes awry: two triangles just can't be three, and neither number is five. (It's only blind luck that two and three make five. In many cases, there's no easy relationship like this.) The process looks discontinuous — there's a saltation: three triangles are two and two are three — but it's not. Shapes can always be described in alternative ways, and welcome different descriptions indifferently. The parts I see may alter erratically either in number or by kind at anytime.

I can describe the shapes in the series as topologies, so that the rotation from the initial shape to the final shape is continuous. This also shows how the shapes are divided into constituents as a result of applying the rule to calculate. In particular, there are Boolean algebras for the shapes with the atoms given in Table 2.

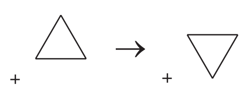
Table 2

Constituents Defined by Calculating

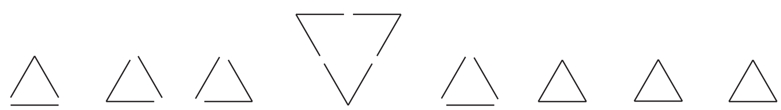


The atoms are defined retrospectively. To quote Kierkegaard — via James — ‘We live [calculate] forwards, but we understand backwards.’ I begin with the final shape and then work back to the initial shape. The final shape has no noticeable structure. It’s an atom itself as it’s not divided by the rule. But each preceding shape has the triangle resolved by the rule as an atom, and other atoms to form its complement and to ensure continuity going forward.

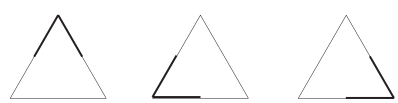
This probably sounds a little obscure — I know I mentioned Kierkegaard — but really I’m simply calculating again to explain how I’ve been calculating. And the idea is clear once it’s tried. Its value as an account of how I calculate is easy to see if I record the divisions in the triangle in the right side of the rule



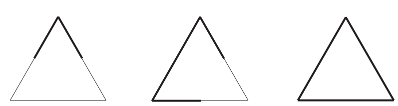
formed with respect to constituents (atoms) in Table 2. The triangle is divided in alternative ways in the eight steps of the series



The parts defined in the triangle combine sequentially to build up the different triangles that the rule picks out in its subsequent applications. The pieces



from the first three triangles combine in this way

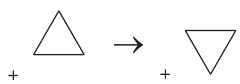


and the remaining sides

combine in this way

to define constituents of the second and third shapes in the series, and at last the large outside triangle and the small inside one in the fourth shape. And the pieces in the fourth and fifth triangles combine in like fashion to make the three triangles in the sixth shape that are needed for the production of the final shape. Looking forward, the constituents of the shapes in the series anticipate what the rule is going to do the next time it's applied. But this isn't divination. It happens because constituents are only given as an afterthought. Whenever calculating stops, I can describe what's gone on as a continuous process in which shapes are assembled piece by piece in an orderly way. This makes a good story and a good explanation. It's the kind of retrospective narrative I hear all of the time from people doing creative work.

Every time the rule



is tried, its right side is divided with respect to different constituents. The way I describe what I'm doing changes as I calculate — it's merely an artifact of what I'm trying for the time being — so I can always go on and calculate some more. Nothing prevents me from seeing things again. I'm free to fuse old divisions and make new ones.

How descriptions of shapes change is highlighted in another way when I use schemas to define rules. A rule schema

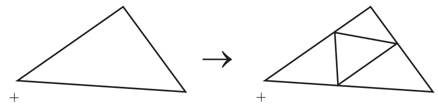
$$x \rightarrow y$$

is a pair of variables x and y that take shapes as values. These are given in an assignment g that satisfies a predicate. Whenever g is used, a rule

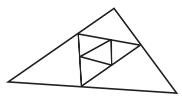
$$g(x) \rightarrow g(y)$$

is defined. The rule then applies as usual, as if it were given explicitly from the start. This works to express a host of intuitive ideas about shapes and how to change them.

Schemas can be very general. In fact, I can give one for all rules. I only have to say that x and y are shapes. But usually schemas have a more refined purpose. Here's a rule



defined in a schema x → y that produces the shape

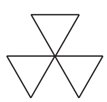


and others of the same kind formed when polygons are inscribed in polygons. An assignment g gives values to the variables x and y that fulfill this predicate:

> x is a polygon with n sides, and y = x + z, where z is a polygon with n sides inscribed in x.

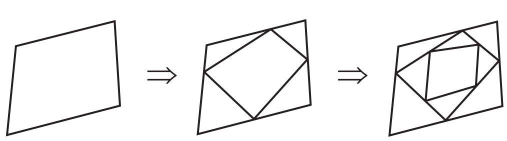
The predicate can be elaborated in much greater detail — for example, I could say something more about polygons being convex, etc. and then go on to say that the vertices of z are points on the sides of x, etc. — but this isn't necessary to see how schemas work. It's enough to have the two variables x and y, and the assignment g.

It's interesting to note also that the schema produces shapes that can be used in examples like the one above in which the shape



is rotated by rotating individual triangles. (The different things that can happen vary with the Fibonacci number determined by the number of polygons that are inscribed one in another.) This shows, too, that my predicate is far from unique. In fact, it's true for any predicate — and in particular, for any predicate for shapes with basic elements of dimension one or more — that there are indefinitely many others equivalent to it.

The specific alternative I have in mind should be obvious. I can change the predicate, so that y is n triangles rather than two polygons. This determines exactly the same rules, but with some interesting twists. In this series of shapes



I use the schema to change a quadrilateral after applying the schema to replace a quadrilateral with triangles. I go from four triangles to a quadrilateral and four smaller triangles by changing a quadrilateral that's inscribed in a quadrilateral. Where did the quadrilaterals come from? I thought I had four triangles. I said it before. The parts I see aren't fixed. They may change at anytime in number or by kind. How I describe a rule and the shape to which it applies needn't agree. I can fool around with descriptions as long as I like, to get the results I want in a sensible way. Rules apply to shapes and not to their descriptions.

This sounds all wrong. How can calculating be so confused? Definite descriptions define the shapes in a rule $g(x) \rightarrow g(y)$. But the description of $g(x)$ may be incompatible with the description I have for the shape to which the rule applies. It's just not calculating if these descriptions don't match. But what law says that shapes must be described consistently to calculate. The shape $g(x)$ may have indefinitely many descriptions besides the description that defines it. But none of these descriptions is final. The embedding relation works for parts of shapes — not for descriptions of shapes — when I try the rule. My account of how I calculate may use as many descriptions as I like that jump from here to there erratically. It may sound nuts or irrational while I'm doing it. But whether or not there's a favorable conclusion with useful results doesn't depend on this. I can always tidy up after I've finished calculating, so that I have a retrospective explanation that's consistent. Rationality is a sentiment to end with.

So when is calculating visual? I have another answer that may be better than my formula dim(el) = dim(em).

Calculating is visual when descriptions don't count.

Descriptions aren't binding. There's no reason to stick with any of them that's not just prejudice. I'm happy with this, but I'm not sure everyone will be happy with what it does to reasoning. I use rules to calculate, but I don't have to play by them. I can cheat. And I can get away with it. I'm totally free to change my mind about what there is, and I'm free to act on it. (Children play games in this way, unless we intervene and make them follow the rules. Only suppose they already do.) There are no definitions to conform to, and there's no vocabulary to build from. Constituents, atoms, components, primitives, units, and the rest are merely afterthoughts. But is this going too far? What makes me think I'm calculating? And if I'm not, what's reasoning all about? Visual calculating is crazy.

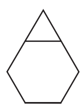
What's that or how many?

I confess. I really don't know what reasoning is, and I'm not totally sure about calculating. I don't think James would have any problem with what I've been saying, either as calculating or as reasoning. But others may demur for one reason or another. Not everyone is ready and willing to allow as much as James — to encourage different points of view and to welcome the novelty they bring — when there's thinking to do. There's always a right way and a wrong way, no doubt about it.

> All ways of conceiving a concrete fact, if they are true ways at all, are equally true ways. *There is no property* ABSOLUTELY *essential to any one thing*. The same property which figures as the essence of a thing on one occasion becomes a very inessential feature upon another.
>
> Meanwhile the reality overflows these purposes at every pore.
>
> . . . *the only meaning of essence is teleological* . . . *classification and conception are purely teleological weapons of the mind*.
>
> The properties which are important vary from man to man and from hour to hour.
>
> Reasoning is always for a subjective interest, to attain some particular conclusion, or to gratify some special curiosity. It not only breaks up the datum placed before it and conceives it abstractly; it must conceive it *rightly* too; and conceiving it rightly means conceiving it by that one particular abstract character which leads to the one sort of conclusion which it is the reasoner's temporary interest to attain.

I may change occasions more rapidly than James expects moving freely among parts whether they're details or overall features. I change occasions every time I apply a rule $M \rightarrow P$ to calculate: when I embed M and infer (introduce) P. And I string all of these occasions together in an ongoing series to produce useful but not necessarily logical (consistent) results. Nothing says my temporary interests aren't evanescent and arbitrarily linked. I think this is why visual calculating and reasoning can be a lot more effective in practice than calculating and reasoning in some other way. Their appeal to seeing — to sensible concrete experience — puts the ability to deal with novelty — '*the technical differentia of reasoning*' — at the center of creative activity.

Some other famous thinkers, however, aren't so sure that my kind of calculating with shapes — where rules automatically redefine parts as they apply — is really calculating. Ludwig Wittgenstein notices that numbers and shapes don't add up in the same way — there's a distinct difference between calculating and visual calculating — and he suggests his observation shows how 'mathematics is *normative*'.

> An addition of shapes together, so that some of the edges fuse, plays a very small part in our life. — As when
>
>  and 
>
> yield the figure
>
> 
>
> But if this were an *important* operation, our ordinary concept of arithmetical addition would perhaps be different.

> Let us imagine that while we were calculating the figures on paper altered erratically. A 1 would suddenly become a 6 and then a 5 and then again a 1 and so on. And I want to assume that this does not make any difference to the calculation because, as soon as I read a figure in order to calculate with it or to apply it, it once more becomes the one that we have in *our* calculating. At the same time, however, one can see quite well how the figures change during the calculations; but we are trained not to worry about this.
>
> Of course, even if we do not make the above assumption, this calculation could lead to usable results.
>
> Here we calculate strictly according to rules, yet this result does not *have* to come out. — I am assuming that we see no sort of regularity in the alteration of figures.

> Now you might of course say: "In this case the manipulation of figures according to rules is not calculation."

Wittgenstein's curious manipulations in which figures alter erratically of their own accord and calculating with shapes have a common look and feel. It's uncanny. And just as the one may skirt the comfortable norms of calculating, so too may the other. Of course, norms may be effective or not. At first, Wittgenstein is tempted to limit what can change and simply go on with business as usual. He's surprisingly explicit about this.

> And I want to assume that this does not make any difference to the calculation because, as soon as I read a figure in order to calculate with it or to apply it, it once more becomes the one that we have in *our* calculating.

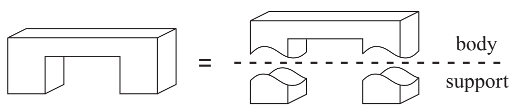
Wittgenstein's instinctive response is conservative. He keeps to the law that keeps everything the same, the same as the rest of us. I count on figures to behave themselves when I use them to calculate. This comes from training. (There are the basics — predefined constituents and their combinations — tough standards, and tests to make sure that schools and students are accountable for teaching and learning. How basic are basics that have to be taught and tested? The need for standards is to exclude anything different. They limit present experience to what seemed to work in the past. Any prospect of novelty is gone. There are no surprises. Nothing is ambiguous or vague. This is the end of anxiety and uncertainty, and makes it unnecessary to trust others and give them a chance. But is it a sensible way to educate people to recognize and exploit new opportunities? What good are my rules now? Of course, training needn't limit experience. It may be open-ended — for example, in studio instruction and situated learning. In the latter, master and apprentice interact during actual practice. They work on the same thing without having to see it in the same way. There's no underlying structure controlling the process: 'structure is more the variable outcome of action than it's invariant precondition.' This is like visual calculating.) Yet rigorous training may not be necessary for a successful (creative or novel) conclusion when I calculate:

> . . . even if we do not make the above assumption, this calculation could lead to usable results.

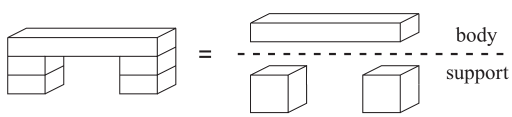
Ignoring 'the above assumption' is what visual calculating is all about. Whether or not you think it's real calculating doesn't really matter. But the ambiguity cuts in opposing ways: once to explain why greater attention hasn't been focused on visual calculating as a useful alternative to calculating with numbers — the one isn't calculating — and then again to explain why it's so easy to think that visual calculating is necessarily the same as calculating in the ordinary way. I like to calculate by seeing. But whenever I try, either no one believes it or they think I'm doing something else. Perhaps I'm just wasting my time when it comes to visual calculating. Only what else can I do if I want to see how far reasoning goes. There's always something else to see.

I haven't clarified anything yet. Calculating and reasoning — visual or not — appear no better off than shapes. They're just as vague and ambiguous. But others have thought about this, and draw distinctions in many alternative ways. Everybody knows that Marvin Minsky — Evans was his student at MIT — thinks calculating and reasoning are the same thing. What he says about creativity is telling.

> What is creativity? How do people get new ideas? Most thinkers would agree that some of the secret lies in finding "new ways to look at things." We've just seen how to use the Body-Support concept to reformulate descriptions of some spatial forms . . . let's look more carefully at how we made those four different arches seem the same, by making each of them seem to match "*a thing supported by two legs*." In the case of *Single-Arch*, we did this by imagining some boundaries that weren't really there: this served to break a single object into three.



> However, we dealt with *Tower-Arch* by doing quite the opposite: we treated some real boundaries as though they did not exist:



> How cavalier a way to treat the world, to see three different things as one and to represent one thing as three!

Minsky echoes James with a neat example. In fact for James, 'Genius, in truth, means little more than the faculty of perceiving in an unhabitual way.' Yes, Minsky welcomes alternative descriptions and knows just how daring (cavalier) an idea this is. But he may be uneasy about it. Elsewhere, he's ready to apply 'radically different kinds of analysis to the same situation' and is convinced that Thomas Kuhn's paradigm switches occur again and again in everyday thought. This still sounds a lot like James. Only Minsky shies and retreats to the security of constituents and combinations. He splits thinking in two: an initial analysis of 'elements' — how is this done? — and an independent heuristic search in which 'elements are combined in different ways until a configuration is reached that passes some sort of test.' This is how computer models are supposed to work, and it's not too far off the dull sort of education

I just described. How useful is analysis before search begins? What happens when rules apply to shapes? There's a paradigm (gestalt) switch every time a rule is tried. The elements of analysis aren't given beforehand but change as calculating goes on. Minsky's arches support this. They furnish the standard that shows what it means to be static, and much more. The statics of arches — 'the Body-Support concept' — works consistently in different cases because arches aren't static. Shapes are never static. They're unstable. They fuse in combination, so that new divisions are always possible. Elements, configurations, and tests aren't the way to deal with this kind of ambiguity. How well does Evans's grammar work? There's an initial analysis to define constituents followed by a search in which a heuristic — try configurations with three lines — and a test — is it a polygon? — are involved. Everything is askew if lines are taken for points that can be counted. Heuristic search is senseless once analysis — seeing — stops.

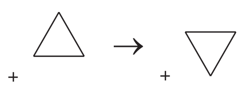
Minsky starts out in the right direction only to make a wrong turn. He's lost. He's sure quantitative models of reasoning are inadequate. This is fine. Visual calculating — everybody knows it's qualitative — may be an alternative worth searching for. Yet Minsky's reason to find something new is baffling. A number-like magnitude 'is too structureless and uninformative to permit further analysis . . . *A number cannot reflect the considerations that formed it*.' OK, but this is also true for shapes. It lets me divide them freely when I apply rules. The opposite assumes a law of conservation — Wittgenstein shows this — to uphold decisions I've made in the past, to recognize (remember) what I did before and act on it heedless of anything else that may come up. (Memory isn't something to forget when buying a computer or thinking about thinking. Plato considers memory a good source of ideas. Suppose I have a computer with the memory to store all of the information I might ever use — important facts and right definitions of things like triangles. Would it help me think? It may take care of learning — there's no reason not to be optimistic — but there's still sagacity — raw creativity. Who wants to visit www.plato.edu or invest in one of its .com counterparts?) This looks away from visual calculating to calculating in the way we've been trained. I'm stuck. Visual calculating isn't calculating or it's misunderstood.

The idea that I can change how I describe things when I apply rules is at the heart of visual calculating, but it seems it's an idea that's not easy to accept or to use. And in fact, it would be easy to dismiss if it only had roots in sensible experience that's superficial and without any deep structure. Art and such are important, but science is what really counts. It's lucky that things aren't always how they appear at first. It's an idea strongly rooted in science as well. Hilary Putnam tells the story.

> Since the end of the nineteenth century science itself has begun to take on a 'non-classical' — that is, a non-seventeenth-century-appearance. [Earlier] I described the phenomenon of conceptual relativity — one which has simple illustrations, like [mine for a few individuals], but which has become pervasive in contemporary science. That there are ways of describing what are (in some way) the 'same facts' which are (in some way) 'equivalent' but also (in some way) 'incompatible' is a strikingly non-classical phenomenon. Yet contemporary logicians and meaning theorists generally philosophize as if it did not exist.

This contains a good description of the shapes in the series

that are produced when I use the rule



to calculate. In particular, the fourth shape and the sixth shape are equivalent — they're congruent — but are incompatible with respect to their parts. The fourth shape is two nested triangles: that's how I get to the sixth shape. And the sixth shape is three corner triangles: that's how I get to the final shape. Indeed, this is a striking way to calculate. That it's outside of the interests of logic is no big surprise. I said at the beginning that I wasn't too fond of logic and would try to avoid it. This only settles my decision. Others who have thought about visual reasoning — for instance, Susanne Langer — look at logic in roughly the same way. (Langer distinguishes 'presentational' and 'discursive' forms. The one deals with sensible experience, while the other sticks with units (constituents) and their combinations.) But Putnam isn't worried about what logicians ignore. He wants to make sense of this non-classical phenomenon. And what he says resonates with what I've been trying to say about visual calculating.

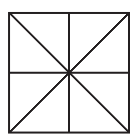
Putnam makes two important points. One works with visual calculating, and the other does — when it goes against convention — and doesn't — when it relies on counting. I'm not concerned if numbers alter without rhyme or reason while I calculate. Putnam's first point is this:

> . . . [the] phenomenon [of conceptual relativity] turns on the fact that *the logical primitives themselves, and in particular the notions of object and existence, have a multitude of different uses rather than one absolute 'meaning'*.

And second,

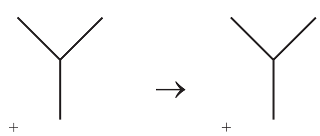
> Once we make clear how we are using 'object' (or 'exist'), the question 'How many objects exist?' has an answer that is not at all a matter of 'convention'.

The first point is pretty obvious whenever I calculate with shapes. What I see before me depends on the rules I try. Evans's shape

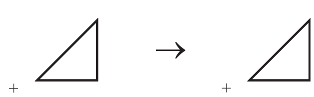


isn't anything in particular until I apply the rule (identity)

to pick out triangles. And if I use the rule

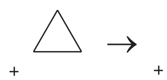


it's something different. Then the second point — at least my version of it — is also clear. It's not up to me if there are triangles in the shape, but the result of calculating with the rule

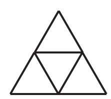


to see how triangles are embedded. The only problem is that the rule can be used in different ways to get different results. Of course, I can always insist that the rule is applied everywhere there's a triangle. And I even have an algorithm for this — in fact, my algorithm works for any rule. So maybe there's a definite answer after all.

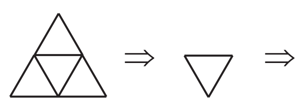
But I have another way to look at counting that gives inconsistent results that aren't so easy to bypass. Suppose I start with the rule



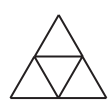
that erases equilateral triangles, and then apply the rule to the shape



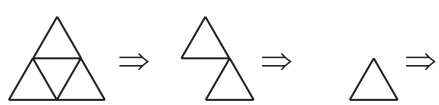
In the series of shapes



there are clearly two triangles. That's precisely how many I erase to make the shape

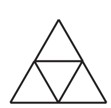


disappear. What better test for counting could I ever have? And in this series

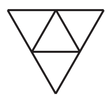


there are three triangles. With only a little fiddling around — rotate the shapes in the second series — it's easy to see that both of the series are implicated in the series of shapes

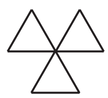
to confirm what I just said that it's a striking way to calculate: the first series in going from the fourth shape



to the sixth shape



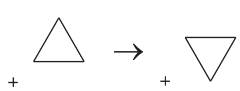
and the second series in going from the sixth shape to the final shape



This is at least the third time — I promise it's the last — that I've shown a variation of the same thing. I must be thinking about my students. I show them how to calculate with shapes again and again. They nod and insist they've got it. Only when it comes to doing it, they do exactly what they're used to. They try constituents, and rely on all of the other bad habits — counting things up, saying they're complex before calculating, etc. — that I'd been trying to break. Jacques Barzun in his nifty book on James calls this the 'rubber-band effect'. I use a few examples — like the ones above — to stretch the rubber-band when I teach. I do this as hard and as often as I can. Once in a while, I'm lucky. The rubber-band snaps. But let's get back to Putnam's second point.

I want to say that the parts a shape has depend on what happens to the shape as I calculate. So long as I continue to try rules, parts may change. But I'd like to be consistent about this whenever I can. If I take the series of shapes

seriously — and I do — then the parts of the shapes are the result of using the rule



in a certain way. Now look at the atoms in Table 2. They take care of the above inconsistencies, so that the series is a continuous process. And I can always do this somehow after I'm through calculating — with topologies, Boolean algebras, lattices of different kinds, or comparable descriptive devices. But more than the variability that's introduced in this way, there are alternative ways of going from the initial shape to the final shape that imply different parts for the shapes in the series. What these parts are isn't a matter of convention, at least in the way that Putnam has in mind. I can't say what the parts are until I calculate. There's real work to do to find out what's what. I have to interact with shapes in the general sort of way I interact with other things like letters or numerals when I read or do arithmetic. No doubt, I can contrive some method to reconcile any differences that arise from calculating in this way or that retrospectively, if it ever becomes useful. Only this may not be the end of it. Whether or not I go on to see other parts as I continue to try rules, there's just no reason to think that differences aren't real.

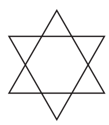
One way to check on experience is to require that parts are numerically distinct: to ask the question how many and get the same answer with the same parts every time. An accountant is trained for this, and a school teacher expects it taking roll. My local Selectman in Brookline is proud of it — 'I'm logical, I'm an accountant'. I know just what she means. I count at the market to make certain I've got all of the items on my grocery list. Counting is an everyday practice that's also good for business and science. But it's only one kind of activity among myriad others. Must everything that counts work this way? What makes counting so special? Wittgenstein muses about shapes when they're added together, so that some of their lines fuse. And James worries if counting is right for sensible experience.

> The relation of numbers to experience is just like that of 'kinds' in logic. So long as experience will keep its kind we can handle it by logic. So long as it will keep its number we can deal with it by arithmetic. *Sensibly,* however, things are constantly changing their numbers, just as they are changing their kinds. They are forever breaking apart and fusing.

Simply counting parts isn't the only way to test experience. I can think of other things to do. One activity is the key to visual calculating. Take your finger and trace the parts you see. Trust your eyes, no matter what parts there are or how often they alter. Who cares whether numbers add up? Rely on the embedding relation. What you see is what you get.

Minsky and Putnam cover some fascinating territory in which things can change. But they shy away from the vast interior. There's novelty there, and the chance to see things in new ways again and again, because they're vague and ambiguous like shapes. Artists, poets, madmen, and perhaps a few pragmatists have ventured into this foreign region with promiscuous results. But no one has seen it all, or ever will. There's always more to see. What's especially nice is to find that visual calculating may be a practical way to map some of the uncertain terrain.

When it comes to handling novelty, I take James as broadly as I possibly can to include seeing new things — visiting Los Angeles for the first time — and equivalently seeing things in new ways — going back (looking again) to find out how the city has changed. In both cases, the same question invokes novel experience. What's that? And of course, this leads to visual calculating, while the above question — how many? — stays with calculating in the normal way. James provides a neat example in *Pragmatism: A New Name for Some Old Ways of Thinking*:

> You can treat [this] figure
>
> 
>
> as a star, as two big triangles crossing each other, as a hexagon with legs set up on its angles, as six equal triangles hanging together by their tips, etc. All these treatments are true treatments — the sensible *that* upon the paper resists no one of them. You can say of a line that it runs east, or you can say that it runs west, and the line *per se* accepts both descriptions without rebelling at the inconsistency.

But why not try to define the options once and for all? What would Evans say about it? I count 24 constituents — six long lines and their thirds — if I'm going to look for triangles. Halves don't work anymore. They were ad hoc from the start. Will the same constituents do the job if I jiggle the two big triangles a little to get another six small ones? What happens if I jiggle harder and harder? Or suppose lines stay put. Then what about diamonds, trapezoids, and the pentagon in Wittgenstein's addition? And what about A's and X's — big ones and little ones? But why should I try to limit my experience before I have it? Isn't that what planning is for? There's always something else to see every time I use another rule — as long as I calculate by seeing.

James and other pragmatists got it right. At least I think so, whenever I calculate with shapes. And today 'neopragmatists' walk James's line that runs east and west. Richard Rorty's ironist is a perfect example as he and she try to redescribe things to make sense of them in a kind of literary criticism instead of philosophy. (This may say a lot more about a professional pecking order than about irony. In Hollywood, writers want to be actors, actors directors,

directors producers, and producers writers. What about philosophers and critics?) The goal isn't coherence but to get around argument — reasoning — 'by constantly shifting vocabularies, thereby changing the subject'. Rorty's ironist always has another verbal trick to see things from a different perspective. It's not a question of truth but 'making things new'. Isn't this embedding all over again? Just how does Rorty put it?

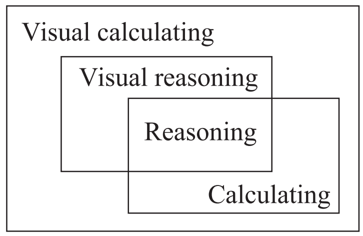
> I have defined 'dialectic' as the attempt to play off vocabularies against one another, rather than merely to infer propositions from one another, and thus as the partial substitution of redescription for inference. I used Hegel's word because I think of Hegel's *Phenomenology* both as the beginning of the end of the Plato-Kant tradition and as a paradigm of the ironist's ability to exploit the possibilities of massive redescription. Hegel's so-called dialectical method is not an argumentative procedure or a way of unifying subject and object, but simply a literary skill — skill at producing surprising gestalt switches by making smooth, rapid transitions from one terminology to another.

The ironist doesn't calculate — at least according to Rorty — but does precisely what I do when I calculate with shapes. We both exploit myriad descriptions in order to do more. It's just like politics. There are no ends — big goals like truth or permanent parts — only means. I apply rules in an algorithmic process, while the ironist uses literary skill to glide effortlessly from one terminology to another in a dialectic. So why the fuss? Nothing stops me from changing descriptions — from being ironic — when I calculate. I've been calculating with shapes for the past 30 years, and this is simply business as usual. There are many twists and turns. Lionel March has followed most of them with me, and is especially good at saying what's involved.

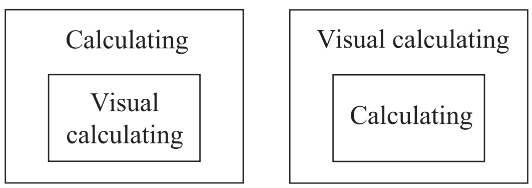
> Contrary to conventional wisdom, rationality does not flourish in the presence of objective certainty, but actually thrives around subjective volition. To be rational requires the willingness to restructure the world on each contingent occasion, or in just two words, TO DESIGN.

And again for me, each contingent occasion is every time I try a rule. Going on means always starting over. There's a new analysis that's independent of any given before. Parts fuse to erase prior distinctions, and the resulting shape divides according to the rule I use now. It's all visual calculating — calculating by seeing. I said at the beginning that I think about this in terms of design. It appears I've been dealing with design all along.

So how are calculating and reasoning, and visual calculating and visual reasoning related? Also at the beginning, I made two suggestions in the form of diagrams. But right now, it's probably a far better bet to rely on the diagram



that changes my first one by reversing the relationship between reasoning and visual calculating. I didn't plan it this way, yet it appears that visual calculating holds more than reasoning allows. Maybe this isn't worth the bother of figuring out. Perhaps the only relationships worth elaborating are these



that chart equivalencies between calculating by counting that asks how many — the ordinary kind of calculating we're used to — and calculating by seeing that asks what's that. How can I finally decide? Well, if calculating and reasoning — whether they're visual or not — are anything at all like shapes, then it's going to depend on how I calculate. The relationships — both what there is and how it's connected — will change as I try rules. We'll have to wait and see how it comes out. What a pleasant way to spend a summer afternoon talking.

Background

The ideas in this paper have secure technical roots. A summary of the main details — including the relationship between calculating by counting and calculating by seeing — can be found in my recent paper 'How to Calculate with Shapes'. It's Chapter 2 in *Formal Engineering Design Synthesis*, edited by Erik Antonsson and Jon Cagan. Cambridge University Press will publish the book later this year (2001). If you have the patience I do, you may want to wait for my book *Shape* to be finished. MIT Press hopes to publish it soon.

All the quotations from James are located in Chapter XXII — 'Reasoning' — in *The Principles of Psychology*, except for the following five. The quotation on numbers and experience is in Chapter XXVIII, and the one on genius is in Chapter XIX. James talks about concepts and perceptual experience near the end of Chapter XI in *Some Problems of Philosophy*. Kierkegaard's saying is found in Chapter IX in *Essays in Radical Empiricism*. And the description of the star is in Lecture 7 in *Pragmatism: A New Name for Some Old Ways of Thinking*.

R. Narasimhan describes Evans's grammar and how it applies to his shape in the first chapter 'Picture Languages' in the book *Picture Language Machines*, edited by S. Kaneff.

The quotation from Wittgenstein appears in section V-40 in *Remarks on the Foundations of Mathematics*. The gloss on situated learning is taken from William F. Hanks's forward to Jean Lave and Etienne Wenger's book *Situated Learning: Legitimate Peripheral Participation*. Hanks's brief description of how portable skills develop 'even when coparticipants fail to share a common code' is also worthwhile. The material from Minsky is in two places. His thoughts on creativity and the Body-Support concept are in section 13.2 in *The Society of Mind*, and the rest is in his chapter 'A Framework for Representing Knowledge' in Patrick Winston's earlier book *The Psychology of Computer Vision*. The first quotation from Putnam is in Lecture 2 in *The Many Faces of Realism*, while the other two are found in Lecture 1. Langer deals with forms of symbolism in *Philosophy in a New Key*, Barzun describes the rubber-band effect in 'The Masterpiece' in *A Stroll with William James*, and Rorty limns the ironist's literary skill in Chapter 4 of *Contingency, Irony, and Solidarity*. Lionel March and I enjoy talking about design whenever we can at the *Moustache Cafe* in West Los Angeles, over a long lunch and a good bottle of wine.

Acknowledgment Mine Ozkar did all of the shapes for me with her usual care and enthusiasm. She also made sure they were in the right places, even as my descriptions of what was going on changed.