Finite Element Methods for Elliptic Problems

Variational Formulation: The Poisson Problem

March 19 & 31, 2003

Motivation

- The Poisson problem has a strong formulation; a minimization formulation; and a weak formulation.

- The minimization/weak formulations are more general than the strong formulation in terms of regularity and admissible data.

Motivation

- The minimization/weak formulations are defined by: a space $\boldsymbol{X}$; a bilinear form $\boldsymbol{a}$; a linear form $\boldsymbol{\ell}$.

- The minimization/weak formulations identify

ESSENTIAL boundary conditions,
Dirichlet — reflected in $\boldsymbol{X}$;

NATURAL boundary conditions,
Neumann — reflected in $\boldsymbol{a}, \boldsymbol{\ell}$.

Motivation

- The *points of departure* for the *finite element method* are:

the weak formulation (more generally);

or

the minimization statement (if a is SPD).

The Dirichlet Problem

Strong Formulation

Find u such that

$$\begin{aligned} -\nabla^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and Ω is a domain in $\mathbb{R}^2$ with boundary Γ.

The Dirichlet Problem

Minimization Principle

Statement...

Find

$$u = \arg\min_{w \in X} J(w)$$

where

N1

$$X = \{v \text{ sufficiently smooth} \mid v|_\Gamma = 0\},$$

and

$$J(w) = \frac{1}{2}\int_\Omega \underbrace{\nabla w \cdot \nabla w}_{w_x^2 + w_y^2}\, dA - \int_\Omega f\, w\, dA\,.$$

N2

The Dirichlet Problem

Minimization Principle

...Statement

In words:

Over all functions w in X,

u that satisfies

$$-\nabla^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \Gamma$$

makes $J(w)$ as small as possible.

N3

The Dirichlet Problem

Minimization Principle

Proof...

Let $w = u + v$.

Then

$$J(\overbrace{\underbrace{u}_{\in X} + \underbrace{v}_{\in X}}^{w \in X}) = \frac{1}{2} \int_\Omega \nabla(u+v) \cdot \nabla(u+v)\, dA$$

$$- \int_\Omega f(u+v)\, dA\,.$$

The Dirichlet Problem

Minimization Principle

...Proof...

$$J(u+v) = \frac{1}{2}\int_\Omega \nabla u \cdot \nabla u \, dA - \int_\Omega f\, u \, dA \qquad J(u)$$

$$+ \int_\Omega \nabla u \cdot \nabla v \, dA - \int_\Omega f\, v \, dA \qquad \delta J_v(u)$$

first variation

$$+ \frac{1}{2}\int_\Omega \nabla v \cdot \nabla v \, dA \qquad > 0 \text{ for } v \neq 0$$

The Dirichlet Problem

Minimization Principle

...Proof...

$$\delta J_v(u) = \int_\Omega \nabla u \cdot \nabla v \, dA - \int_\Omega f \, v \, dA$$

$$= \int_\Omega \nabla \cdot (v \nabla u) \, dA - \int_\Omega v \nabla^2 u \, dA - \int_\Omega f \, v \, dA$$

$$= \int_\Gamma \cancelto{0}{v} \, \nabla u \cdot \hat{n} \, dS + \int_\Omega v \{ \underbrace{-\nabla^2 u - f}_{0} \} \, dA$$

$$= 0 \, , \qquad \forall \, v \in X$$

N4

The Dirichlet Problem

Minimization Principle

...Proof

$$J(\underbrace{u+v}_{w}) = J(u) + \frac{1}{2}\underbrace{\int_{\Omega} \nabla v \cdot \nabla v \, dA}_{>0 \text{ unless } v=0}, \; \forall\, v \in X$$

$\Rightarrow$

$$J(w) > J(u), \qquad \forall\, w \in X,\; w \neq u$$

$$\Updownarrow$$

u is *the* minimizer of $J(w)$

E1

The Dirichlet Problem

Weak Formulation

Statement

Find $u \in X$ such that

$$\delta J_v(u) = 0\,, \quad \forall\, v \in X$$

$$\Updownarrow$$

$$\int_\Omega \nabla u \cdot \nabla v \, dA = \int_\Omega f\, v\, dA\,, \qquad \forall\, v \in X \quad ;$$

see Slide 9 for proof.

N5

The Dirichlet Problem

Weak Formulation

Definitions...

Linear space, $\boldsymbol{Y}$:

A set $\boldsymbol{Y}$ is a linear (or vector) space

if

$$\boxed{\begin{array}{rl} \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{Y}, & \boldsymbol{v}_1 + \boldsymbol{v}_2 \in \boldsymbol{Y} \\ \forall\, \boldsymbol{\alpha} \in \mathbb{R}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{Y}, & \boldsymbol{\alpha v} \in \boldsymbol{Y} \end{array}} \; .$$

The Dirichlet Problem

Weak Formulation

...Definitions...

Linear forms, $L(v)$:

$$L\colon \underbrace{Y}_{\text{input}} \rightarrow \underbrace{\mathbb{R}}_{\text{output}} \qquad \text{(\textit{form} or \textit{functional})}$$

$$L(\alpha v_1 + v_2) = \alpha L(v_1) + L(v_2) \qquad \text{(\textit{linear})}$$

$$\forall\, \alpha \in \mathbb{R}\,, \quad \forall\, v_1, v_2 \in Y\ .$$

The Dirichlet Problem

Weak Formulation

...Definitions...

Bilinear forms, $B(w,v)$:

$B\colon Y \times Z \to \mathbb{R}$ (*form*) ;

$B(w,\bar{v})$ linear form in w for fixed $\bar{v}$,

$B(\bar{w},v)$ linear form in v for fixed $\bar{w}$ (*bilinear*) .

SPD bilinear forms, $B(w, v)$:

$B\colon Y \times Y \to \mathbb{R}$ is *bilinear* ;

$B(w, v) = B(v, w)$ *S*PD ;

$B(w, w) > 0$, $\quad \forall\, w \in Y$, $w \neq 0$ S*PD* .

The Dirichlet Problem

Weak Formulation

Restatement...

Let

$$a(w,v) = \int_{\Omega} \nabla w \cdot \nabla v \, dA\,, \quad \forall\, w, v \in X$$

an *SPD bilinear form*

E2

and

$$\ell(v) = \int_{\Omega} f\, v\, dA, \quad \forall\, v \in X$$

a *linear form* .

The Dirichlet Problem

Weak Formulation

...Restatement

Minimization Principle:

$$u = \arg \min_{w \in X} \underbrace{\frac{1}{2}\, a(w,w) - \ell(w)}_{J(w)} \; .$$

Weak Statement: $u \in X$,

$$\underbrace{a(u,v) = \ell(v)}_{\Leftrightarrow\ \delta J_v(u)=0}, \qquad \forall\, v \in X \; .$$

E3

The Dirichlet Problem

Weak Formulation

Proper Spaces: $u \in X$

Since a involves *only first derivatives* ,

$$X = \{v \in H^1(\Omega) \mid v|_\Gamma = 0\} \equiv H^1_0(\Omega):$$

$$H^1(\Omega) \equiv \{v \mid \int_\Omega v^2 \, dA \, , \int_\Omega v_x^2 \, dA \, , \int_\Omega v_y^2 \, dA \text{ finite}\} \, ;$$

$$\underbrace{(w, v)_{H^1(\Omega)}}_{\text{inner product}} = \int_\Omega \nabla w \cdot \nabla v + w v \, dA \, ;$$

$$\underbrace{\|w\|_{H^1(\Omega)}}_{\text{norm}} = \left(\int_\Omega |\nabla w|^2 + w^2 \, dA \right)^{1/2}$$

N6 E4

The Dirichlet Problem

Weak Formulation

Proper Spaces: $\ell \in X'$

The "data" ℓ: $H_0^1(\Omega) \to \mathbb{R}$ must satisfy

$$|\ell(v)| \le C \, \|v\|_{H^1(\Omega)}, \ \forall \, v \in H_0^1(\Omega) \ (\textit{bounded}).$$

$\ell \in$ *dual space* $X' = (H_0^1(\Omega))' \equiv H^{-1}(\Omega)$:

all linear functionals bounded for $v \in H_0^1(\Omega)$.

Dual norm: $$\|\ell\|_{(H_0^1(\Omega))'} = \sup_{v \in H_0^1(\Omega)} \frac{\ell(v)}{\|v\|_{H^1(\Omega)}}.$$ N7 N8

The Dirichlet Problem

Weak Formulation

Proper Spaces: Well-Posedness

Given $\ell \in H^{-1}(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \ell(v), \qquad \forall\, v \in H_0^1(\Omega)\ .$$

Well-posedness:

u *exists* and is *unique* ; E5 N9

$\|u\|_{H^1(\Omega)} \le C\ \|\ell\|_{H^{-1}(\Omega)}$ — *stability*.

N10 E6 E7

The Neumann Problem

Strong Formulation

Find u such that

$$\boxed{\begin{aligned} -\nabla^2 u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma^D \\ \frac{\partial u}{\partial n} &= g && \text{on } \Gamma^N \end{aligned}}$$

where $\overline{\Gamma} = \overline{\Gamma}^D \cup \overline{\Gamma}^N$, Γ^D non-empty.

N11

The Neumann Problem

Minimization Principle

Statement

Find

$$u = \arg \min_{w \in X} J(w)$$

where

$$X = \{v \in H^1(\Omega) \mid v|_{\Gamma^D} = 0\}$$

$$J(w) = \frac{1}{2} \int_\Omega \nabla w \cdot \nabla w \, dA - \int_\Omega f \, w \, dA - \int_{\Gamma^N} g \, w \, dS \; .$$

The Neumann Problem

Minimization Principle

Proof...

Let $w = u + v$.

Then

$$J(\overbrace{\underbrace{u}_{\in X} + \underbrace{v}_{\in X}}^{w \in X}) = \frac{1}{2}\int_{\Omega} \nabla(u+v)\cdot\nabla(u+v)\, dA$$

$$-\int_{\Omega} f(u+v)\, dA - \int_{\Gamma^N} g(u+v)\, dS\ .$$

The Neumann Problem

Minimization Principle

...Proof...

$$J(u+v) =$$

$$\frac{1}{2}\int_{\Omega} \nabla u \cdot \nabla u \, dA - \int_{\Omega} f \, u \, dA - \int_{\Gamma^N} g \, u \, dS$$

$$+ \int_{\Omega} \nabla u \cdot \nabla v \, dA - \int_{\Omega} f \, v \, dA - \int_{\Gamma^N} g \, v \, dS$$

$$+ \frac{1}{2}\int_{\Omega} \nabla v \cdot \nabla v \, dA$$

The Neumann Problem

Minimization Principle

...Proof...

$$\delta J_v(u) = \int_\Omega \nabla u \cdot \nabla u \, dA - \int_\Omega f\, v\, dA - \int_{\Gamma^N} g\, v\, dS$$

$$= \int_\Omega \nabla \cdot (v \nabla u)\, dA - \int_\Omega v \nabla^2 u \, dA - \int_\Omega f\, v\, dA - \int_{\Gamma^N} g\, v\, dS$$

$$= \int_{\Gamma^D} \cancelto{0}{v} \; \nabla u \cdot \hat{n}\, dS + \int_\Omega v \{ \underbrace{-\nabla^2 u - f}_{0} \}\, dA$$

$$+ \int_{\Gamma^N} v \{ \underbrace{\nabla u \cdot \hat{n} - g}_{0} \}\, dS \quad = \quad 0\,, \qquad \forall\, v \in X$$

The Neumann Problem

Minimization Principle

...Proof

$$J(u+v) = J(u) + \underbrace{\frac{1}{2}\int_{\Omega} \nabla v \cdot \nabla v \, dA}_{> 0 \text{ unless } v = 0}, \; \forall\, v \in X$$

$$\Rightarrow \quad \boxed{\begin{array}{c} J(w) \geq J(u), \qquad \forall\, w \in X; \\ \Updownarrow \\ u \text{ is \textit{the} minimizer of } J(w). \end{array}}$$

E8

The Neumann Problem

Weak Formulation

Statement...

Find $u \in X$ such that

$$\delta J_v(u) = 0\,, \quad \forall\, v \in X$$

$$\Updownarrow$$

$$\int_\Omega \nabla u \cdot \nabla v \, dA = \int_\Omega f\, v\, dA + \int_{\Gamma^N} g\, v\, dS\,, \quad \forall\, v \in X\,;$$

see Slide 25 for proof.

Let:

$$a(w,v) = \int_\Omega \nabla w \cdot \nabla v \, dA \,, \qquad \forall\, w, v \in X$$

bilinear, SPD form ;

and

$$\ell(v) = \int_\Omega f\, v\, dA + \int_{\Gamma^N} g\, v\, dS$$

linear, bounded form (in $H^{-1}(\Omega)$) .

The Neumann Problem

Weak Formulation

...Statement

Minimization Principle:

$$u = \arg \min_{w \in X} \underbrace{\frac{1}{2} a(w,w) - \ell(w)}_{J(w)} \; .$$

Weak Statement: $u \in X$,

$$\underbrace{a(u,v) = \ell(v)}_{\Leftrightarrow \; \delta J_v(u) = 0}, \qquad \forall \, v \in X \; .$$

The Neumann Problem

Weak Formulation

Essential vs. Natural

Essential boundary conditions: Imposed by $\boldsymbol{X}$.

Natural boundary conditions: Imposed by $\boldsymbol{J}$ (or $\boldsymbol{a}, \boldsymbol{\ell}$).

Here:

Essential $\Leftrightarrow$ Dirichlet $(v|_{\Gamma^D} = 0)$,

Natural $\Leftrightarrow$ Neumann $(v|_{\Gamma^N}$ unrestricted$)$.

N12

Important theoretical and numerical ramifications.

E9 E10 E11

Inhomogeneous Dirichlet Conditions

Strong Formulation

Find u such that

$$
\begin{aligned}
-\nabla^2 u &= f && \text{in } \Omega \\
u &= u^D && \text{on } \Gamma^D = \Gamma\,;
\end{aligned}
$$

simple extension to mixed Neumann or Robin.

Inhomogeneous Dirichlet Conditions

Minimization Statement

Find

$$u = \arg \min_{w \in X^D} J(w)$$

where $X^D = \{v \in H^1(\Omega) \mid v|_{\Gamma^D} = u^D\}\ ,$

$$X = \{v \in H^1(\Omega) \mid v|_{\Gamma^D} = 0\}\ ,$$

$$J(w) = \frac{1}{2} \underbrace{\int_\Omega \nabla w \cdot \nabla w \, dA}_{a(w,w)} - \underbrace{\int_\Omega f\, w \, dA}_{\ell(w)}\ .$$

Inhomogeneous Dirichlet Conditions

Weak Formulation

E12

Find $u \in X^D$ such that

$$\boxed{\delta J_v(u) = 0\,, \qquad \forall\, v \in X \equiv H_0^1(\Omega)}$$

$$\Updownarrow$$

$$\boxed{\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dA}_{a(u,v)} = \underbrace{\int_\Omega f\, v \, dA}_{\ell(v)}\,, \qquad \forall\, v \in X\,.}$$

Summary

- The Poisson problem has a strong formulation; a minimization formulation; and a weak formulation.

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Summary

- The minimization/weak formulations are defined by: a space X; a bilinear form a; a linear form ℓ.

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  or
  the minimization statement (if a is SPD).