Fourier analysis of discretized PDEs

PDE

$$\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} = 0$$

Periodic: $U(x+nL,t) = U(x,t)$
for any integer n

Substitute a Fourier mode:

$$U(x,t) = \hat{U}_m(t)\exp(ik_m x)$$

$k_m = \frac{2\pi m}{L}$ for any integer m

$$\frac{d\hat{U}_m}{dt}\exp(ik_m x) + ik_m c\,\hat{U}_m(t)\exp(ik_m x) = 0$$

Semi-discrete w/central space

$$\frac{dV_j}{dt} + c\frac{V_{j+1}-V_{j-1}}{2\Delta x} = 0$$

Periodic: $V_{j+n\frac{L}{\Delta x}}(t) = V_j(t)$
for any integer n

Substitute a Fourier mode:

$$V_j(t) = \hat{V}_m(t)\exp(ik_m x_j) = \hat{V}_m(t)\exp(ik_m j\Delta x)$$

$$\frac{d\hat{V}_m}{dt}\exp(ik_m j\Delta x) + \frac{c}{2\Delta x}\hat{V}_m(t)\left[\exp(ik_m(j+1)\Delta x) - \exp(ik_m(j-1)\Delta x)\right] = 0$$

$$\frac{d\hat{U}_m}{dt} = \underbrace{-ik_m c}_{\lambda_U} \hat{U}_m$$

$$\frac{d\hat{V}_m}{dt} = -\frac{c}{2\Delta x}\left[\exp(ik_m\Delta x) - \exp(-ik_m\Delta x)\right]\hat{V}_m$$

$$\frac{d\hat{V}_m}{dt} = \underbrace{-i\frac{c}{\Delta x}\sin(k_m\Delta x)}_{\lambda_V}\hat{V}_m$$

Comment: For $k_m\Delta x \ll 1$ (wave has long wavelength compared to Δx)

$$\lambda_V \approx -i\frac{c}{\Delta x}(k_m\Delta x) = -ick_m = \lambda_U$$

Discrete eigenvalues converge to PDE eigenvalues as $\Delta x \to 0$.

wavelength

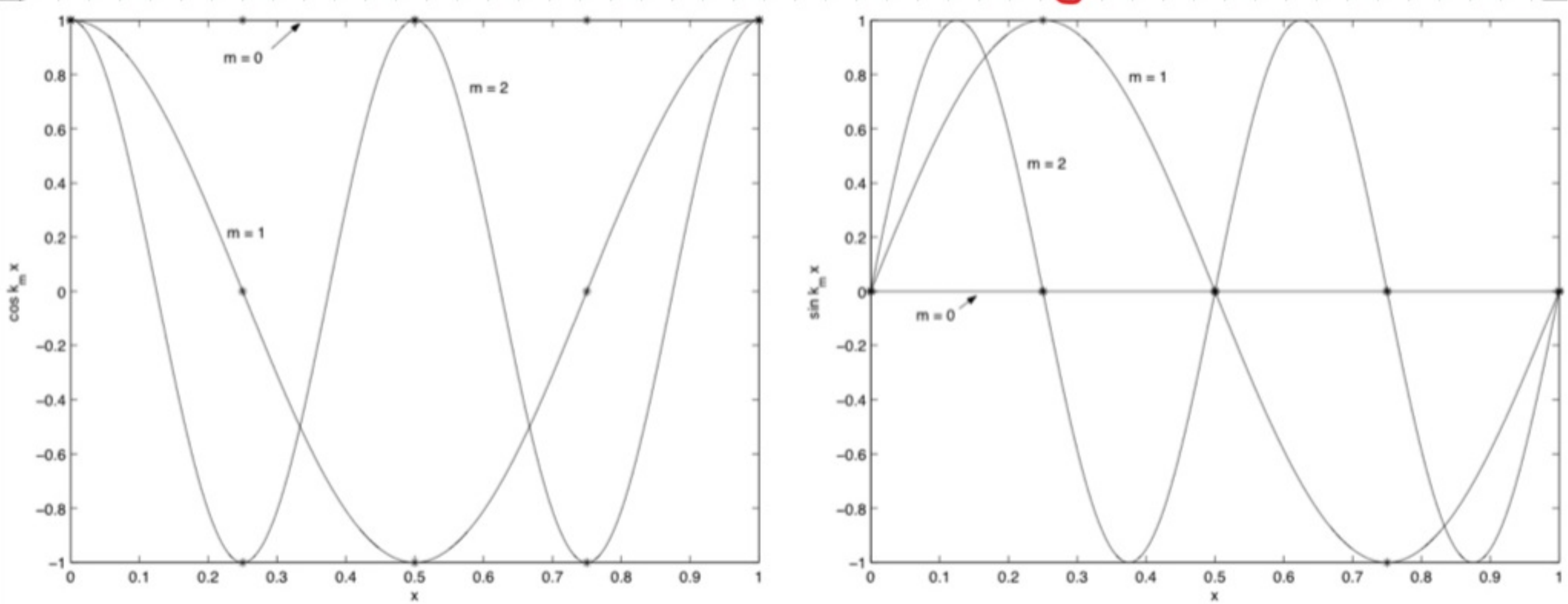
How does $e^{ik_m x_j}$ behave? $k_m = \frac{2\pi m}{L} \Rightarrow l_m = \frac{2\pi}{|k_m|} = \frac{L}{|m|}$

When $l_m < 2\Delta x$, aliasing!

Example: $L = 1$, $\Delta x = 1/4$

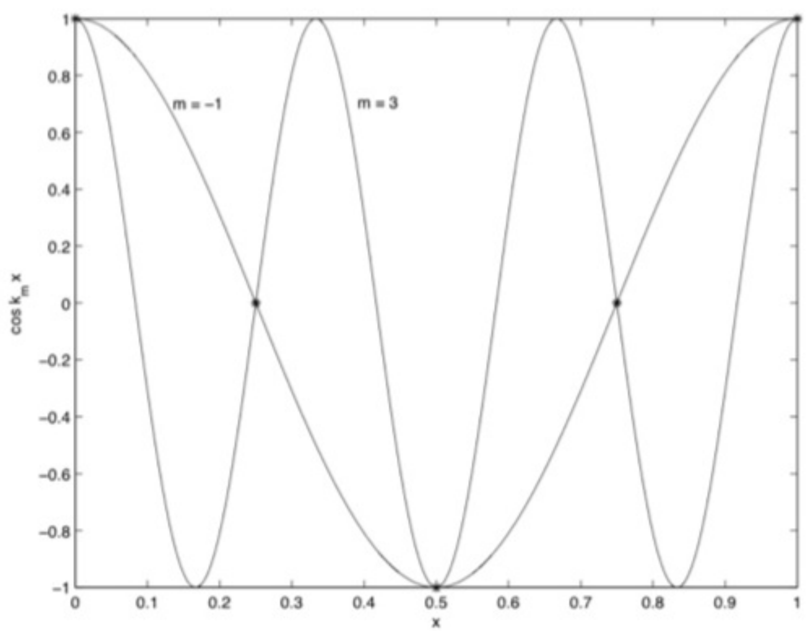
Equivalently when $|k_m| \Delta x > \pi$ aliasing occurs!

Note: $\frac{l_m}{\Delta x} = \frac{L}{|m| \Delta x}$



(a) Real $e^{ik_m x} = \cos k_m x$

(b) Imag $e^{ik_m x} = \sin k_m x$



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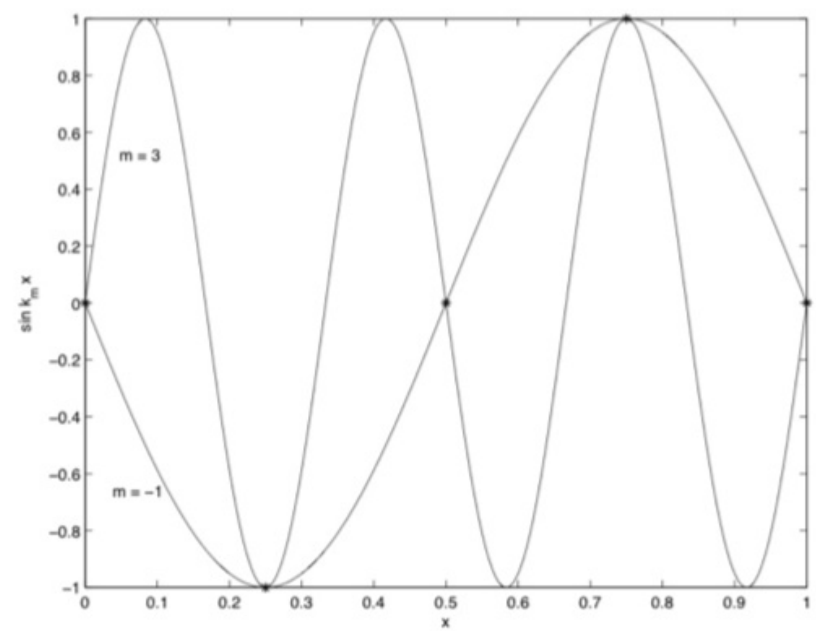


(b) Imag $e^{ik_m x} = \sin k_m x$

$$-\pi \le \beta_m \equiv k_m \Delta x \le \pi$$

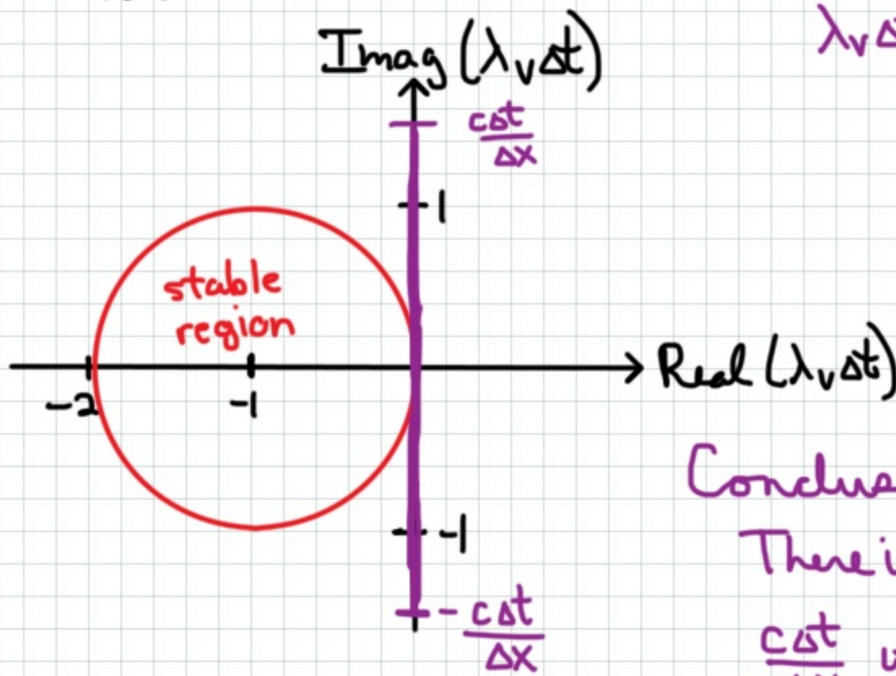
Now, let's return to our semi-discrete analysis:

$$\frac{d\hat{V}_m}{dt} = \lambda_v \hat{V}_m \qquad \text{where } \lambda_v = -i\frac{c}{\Delta x}\sin\beta_m$$

Determine stability for a specific time integration method:

Criterion: $\lambda_v \Delta t$ must be in stability region for chosen time integration.

Example: Forward Euler:



$$\lambda_v \Delta t = -i\,\frac{c\Delta t}{\Delta x}\sin\beta_m$$

Conclusion:

There is no value of $\frac{c\Delta t}{\Delta x}$ which will be stable.

Summary of Fourier analysis of stability:

(1) Substitute $v_j(t) = \hat{v}_m(t) \exp(ij\beta_m)$ into semi-discrete method

(2) Determine eigenvalue $\lambda_v(\beta_m)$ such that $\frac{d\hat{v}_m}{dt} = \lambda_v \hat{v}_m$

(3) Determine timestep requirements for all $\lambda_v(\beta_m)\Delta t$ to be inside stable region for a chosen time integrator

Example: Diffusion: $\frac{\partial U}{\partial t} = \mu \frac{\partial^2 U}{\partial x^2}$ for $\mu > 0$

Spatial discretization: centered, second order

Time integrator: Forward Euler

Determine Δt requirements for stability