Radiation Transport in a Gas (Continued)

<u>Semi-Thick Plasmas Diffusion Approximation</u>

An LTE plasma is such that its collisional processes are fast and equilibrate, but coupling to radiation is incomplete. Then, as we saw, $\vec{\Omega}\cdot\nabla I_\nu = k'_\nu(B_\nu - I_\nu)$. The quantity $\int_0^L k'_\nu d\varepsilon$ is called the optical depth, and if it is large, then $I_\nu \simeq B_\nu$ and we have black-body radiation as well, e.g., full equilibrium. In many cases, plasmas (or other bodies) are not quite so <u>thick</u> (optically); the next approximation is simply $I_\nu \simeq B_\nu - \frac{1}{k'_\nu}\vec{\Omega}\cdot\nabla B_\nu$ which is still semi-local. This is the <u>diffusion approximation</u>.

To calculate energy flux at a point, we need to look at all $\vec{\Omega}$ directions

$$\vec{s} = \int_{4\pi} I_\nu \vec{\Omega} d\vec{\Omega}$$

and then $\nabla\cdot\vec{s}_\nu$ measure the net local cooling rate by radiation. In the diffusion approximations,

$$\vec{s}_\nu = \cancel{\int_{4\pi} B_\nu \vec{\Omega} d\vec{\Omega}} - \int_{4\pi} \frac{1}{k'_\nu} \underbrace{(\vec{\Omega}\cdot\nabla B_\nu)}_{|\nabla B_v| cos\theta} \underbrace{\vec{\Omega}}_{cos\theta\ sin\theta cos\varphi\ sin\theta sin\varphi} \underbrace{d\vec{\Omega}}_{sin\theta d\theta d\varphi}$$

Only the component along ∇B_ν survives $\rightarrow$ $\quad \vec{s}_\nu = -\frac{\nabla B_\nu}{k'_\nu} 2\pi \underbrace{\int_0^\pi cos^2\theta sin\theta d\theta}_{\frac{2}{3}}$

Or, since $u_\nu = \frac{4\pi}{c}B_\nu$, $\vec{s}_\nu = -\frac{c}{3}\frac{\nabla u_\nu}{k'_\nu}$. This should be OK near the center of resonant lines.

This is for one frequency. For all frequencies, $\vec{s} = \int_0^\infty \vec{s}_\nu d\nu$. Using

$$u_\nu = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT}-1} \quad , \quad \nabla u_\nu = \frac{8\pi h\nu^3}{c^3}\frac{e^x \frac{h\nu}{kT^2}}{(e^x-1)^2}\nabla T \quad \left(x = \frac{h\nu}{kT}\right)$$

Then, integrating, $\boxed{\vec{s} = \frac{16}{3}\langle l_\nu\rangle \sigma T^3 \nabla T}$, $\boxed{l_\nu = \int_0^\infty \frac{15}{4\pi^4}\frac{1}{k'_\nu}\frac{x^4 e^{-x}}{(1-e^{-x})^2}dx}$

For $k'_\nu = \text{const.}$, $\langle l_v\rangle = \frac{1}{k'_\nu}$, otherwise it is sensitive to T.

Very often plasmas are only thick near the centers of strong absorption lines, and are fairly <u>thin</u> in between. Let us examine the nature of k_ν.

Bound-Bound electronic transitions. Line radiation. Broadening.

The cross-section for absorption of a photon of $h\nu = E_m - E_n$ by an n-level atom is expressed as an integrated value (over frequency) times a shape factor $\phi(\nu)$:

$$\alpha_{nm} = \boxed{Q_{\nu_n m} = \underbrace{\frac{e^2}{4\epsilon_0 m_e c}}_{2.65\times 10^{-6} m^2/sec} f_{nm}\phi(\nu)} \qquad \int_0^\infty \phi(\nu)d\nu \equiv 1$$

where f_{nm} is the absorption oscillator strength (non dimensional, between 0 and 1). The oscillator strength is mostly empirical, but it is well known for many transitions. For the H atom (and roughly for all alkalis),

$$f_{nm} \simeq \frac{1.96 g_{bb}}{n^5 m^3 \left(\frac{1}{n^2} - \frac{1}{m^2}\right)^3} \qquad (g_{bb} \sim 1, \text{especially at high } m, n)$$

From $Q_{\nu_n m}$ we easily calculate absorption coefficients:

$$k_{\nu_n m} = N_n Q_{\nu_n m}$$

The line shape $\phi(\nu)$ integrates to unity by definition: $\int_0^\infty \phi(\nu)d\nu = 1$. For low T, low P, only natural line shape. "Lorentzian", meaning,

$$\boxed{\phi(\nu) = \frac{1}{\pi}\frac{\gamma/4\pi}{(\nu-\nu_c)^2 + (\gamma/4\pi)^2}} \qquad \phi(\nu_c) = \frac{4}{\gamma}$$

where $\nu_c = \frac{E_m - E_n}{h}$ is the line-center frequency and there is a natural line-width at half-maximum of $\Delta\nu_N = \gamma/2\pi$, due to Heisenberg uncertainly in energy of the levels because of their finite lifetime:

$$\gamma = \sum_{j<m} A_{mj} + \sum_{j<n} A_{nj}$$

These γ's and $\Delta\nu_N$ are very small ($\Delta\lambda = \lambda_c \frac{\Delta\nu}{\nu_c} \sim 10^{-4} \mathring{A}$)

Collisions can be seen as modulations on the atomic oscillators, and they therefore create "sidebands", or broadening. Natural collisions produce Lorentz broadening (or pressure broadening) if it is with unlike atoms, and Holtzmark (or resonance) broadening if it is with the same species. Same line shapes as natural, but replace γ by,

$$\begin{aligned} \Gamma &= \gamma + 2\nu^{\text{opt}} \\ \nu^{\text{opt}} &= \sum_p n_p g_{op} Q_{op} \rightarrow \text{(close to (but larger) than cross-section for momentum exchange)} \end{aligned}$$

1 atm, $\Delta\lambda_{\text{Lorentz}} \simeq 0.05\mathring{A}$, so much more important than natural. But tiny at $n\nu 10^{-18} - 10^{20} m^{-3}$.

Collisions with electrons or other charged particles can also produce broadening (Stark broadening) in plasmas. See Griem (1964); can be very strong at $n_e > 10^{22} m^{-3}$. The easiest broadening to understand is the due to the thermal random motion of the atoms, since there is Doppler shifting of the apparent line center depending on line-of-sight velocity. This is the Doppler broadening, and depends only on T:

$$\phi(\nu) = \frac{1}{\sqrt{\pi}\delta} e^{-\left(\frac{\nu-\nu_c}{\delta}\right)^2} \quad ; \quad \delta = \frac{\Delta\nu_D}{2\sqrt{\ln 2}}$$

$$\Delta\nu_D = \nu_c \sqrt{\frac{8kT\ln 2}{m_a c^2}} = 7.16 \times 10^{-7} \nu_c \sqrt{\frac{T}{M_a}}$$

For Na D-lines at $2000^\circ K, \Delta\lambda_D \sim 0.04\mathring{A}$.

Lorentz and Doppler broadening are very often both important. In the center, Doppler dominates, but the center is often black anyway; in the wings, Lorentz dominates. We can combine the Lorentz and Doppler profiles into a Voigt profile:

$$\phi(\nu) = \frac{1}{\sqrt{\pi}\delta} V(a,x) \qquad \delta = \frac{\Delta\nu_D}{2\sqrt{\ln 2}}$$

$$V(a,x) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}dy}{a^2+(x-y)^2} \qquad a = \sqrt{\ln 2}\frac{\Gamma/2\pi}{\Delta\nu_D}$$

$$x = 2\sqrt{\ln 2}\frac{\nu-\nu_c}{\Delta\nu_D}$$

Bound-Free Radiation

A photon can be absorbed by a bound electron which is below ionization by less that $h\nu$. The event causes ionization (photo-ionization). Similar physics applies if $h\nu \gg eV_i$ (essentially the free electron Thompson scattering). The cross-section for a hydrogenic atom from a principal quantum number m is

$$Q_{\nu_{photoioniz}} = 7.91 \times 10^{-22} \frac{m}{z^2} \left(\frac{Em\eta}{h\nu}\right)^3 \underbrace{g_{bf}}_{\simeq 1} \quad (Em\eta < h\nu)$$

where $Em\eta$ is the energy below ionization (threshold for $h\nu$).

Free-Free (Bremsstrahlung)

An electron bouncing off a heavy particle emits radiation due to its acceleration. Conversely, it can absorb a photon while in the vicinity of a heavy particle, which carries away for of the momentum. The coefficient of free-free absorption (which gives the absorption rate when multiplied times the electron density, the heavy particle density and $\frac{I_\nu}{h\nu}d\nu d\vec{\Omega}$, is

$$\beta_{\text{Free-Free}} = 180 \frac{z^2 g_{ff}}{\nu^3 c_e} \quad (m^5) \quad (g_{ff} \simeq 1)$$

where c_e is the electron speed. Integrating for a Maxwellian electron distribution, the effective "cross-section" is

$$Q_{\nu_{\text{Free-Free}}} = 230 \frac{n_e z^2 \bar{g}_{ff}}{\nu^3 \bar{c}_e}$$

Escape factor for Resonant Radiation

Resonant radiation is radiation from $m \rightarrow$ ground transitions. Since ground is strongly populated, resonant radiation is very likely to reabsorb, at least near the line center. So, instead of ..., we probably see .... How much radiation does escape from a give radiating volume? Take a slab geometry, assume uniform properties, $I_\nu(0) = 0$:

$$I_\nu \;=\; B_\nu(1 - e^{-k'_\nu \varepsilon}) \qquad \varepsilon \;=\; \frac{x}{\cos\theta}$$

Call $s = k'_\nu =$"optical depth". Depends both on distance and on where in the spectrum (k'_ν). We are interested in the 1-D "radiant heat flux", $\vec{s}_\nu = \int_\Omega I_\nu \vec{\Omega} d\Omega$

$$(\vec{s}_\nu)_x \;=\; q_\nu \;=\; \int_0^{\pi/2} (I_\nu \cos\theta) 2\pi \sin\theta d\theta \;=\; B_\nu 2\pi \int_0^{\pi/2} (1 - e^{-\frac{s}{(\cos\theta)}}) \cos\theta \sin\theta d\theta$$

$$\text{call } u \;=\; \cos\theta \qquad q_\nu \;=\; 2\pi B_\nu \int_0^1 (1 - e^{-\frac{s}{u}}) u du$$

This can be done is therms of exponential integrals. But for an approximate solution, note So, make,

$$(1 - e^{-\frac{s}{u}})u \simeq \begin{cases} u & (u < s) \\ s & (1 > u > s) \end{cases}$$

Near,

$$\begin{aligned} q_\nu \;&\simeq\; 2\pi B_v \Bigg[\int_0^s u du + \int_s^1 s du \Bigg] \\ &=\; 2\pi B_\nu \left(\frac{s^2}{2} + s(1 - 2) \right) \\ &=\; 2\pi B_\nu s \left(1 - \frac{s}{2} \right) \qquad (s \ll 1) \\ q_\nu \;&\simeq\; 2\pi B_\nu \int_0^1 u du \;=\; \pi B_\nu \qquad (s \gg 1) \end{aligned}$$

So,

$$\frac{q_\nu}{\pi B_\nu} \simeq \begin{cases} s(2 - s) & s \ll 1 \\ 1 & s \gg 1 \end{cases}$$

Now, assume line is Pressure broadened, at least in the wings:

$$k_\nu \;=\; \frac{k_\nu(\nu_c)}{1 + \left(2\frac{\nu - \nu_c}{\Delta\nu} \right)^2} \qquad (\Delta\nu = \text{full width at } 1/2 \text{ height})$$

$$\rightarrow s \;=\; \frac{A}{1 + \delta^2} \qquad \left(A \;=\; L k_\nu(\nu_c) \quad , \quad \delta \;=\; 2\frac{\nu - v_c}{\Delta\nu} \right)$$

$$\delta \;=\; \sqrt{\frac{A}{s} - 1}$$

Re-plotting $q_\nu/\pi B_\nu$ vs. frequency now,

$$\frac{q_\nu}{\pi B_\nu} = \begin{cases} \frac{A}{1+\delta^2}\left(2-\frac{A}{1+\delta^2}\right) & |\delta| > \sqrt{A-1} \\ 1 & |\delta| < \sqrt{A-1} \end{cases}$$

when $s=1$, $\delta = \sqrt{A-1}$ $\quad$ ($A > 1$ assumed)
Then, for all frequencies in the line,

$$q \;=\; \int_0^\infty q_\nu d\nu \;=\; \frac{\Delta\nu}{2}\int_{-\infty}^\infty g_\nu d\delta \;=\; \frac{\Delta\nu}{2}\pi B_\nu 2\left\{\int_0^{\sqrt{A-1}} 1 d\delta + \int_{\sqrt{A-1}}^\infty s(2-s)d\delta\right\}$$

In general,

$$\delta \;=\; \sqrt{\frac{A}{s}-1} \qquad d\delta \;=\; \frac{-A/s^2 ds}{2\sqrt{\frac{A}{s}-1}}$$

$$\int_{\sqrt{A-1}}^\infty (2s-s^2)d\delta \;=\; -\int_1^0 (2s-s^2)\frac{A/s^2}{2\sqrt{\frac{A}{s}-1}}ds \;=\; A\int_0^1 \frac{2-s}{2\sqrt{A-s}}\frac{ds}{\sqrt{s}}$$

For cases of interest $A = zk_\nu(\nu_c) \gg 1$, while s here is in (0-1), so $\sqrt{A-s} \simeq \sqrt{A}$

$$\int_{\sqrt{A-1}}^\infty (2-s^2)d\delta \simeq \frac{A}{2\sqrt{A}}\int_0^1 (2-s)\frac{ds}{\sqrt{s}} \;=\; \frac{\sqrt{A}}{2}\left(4\sqrt{s}-\frac{2}{3}s^{3/2}\right)_0^1 \;=\; \frac{5\sqrt{A}}{3}$$

Also, $\quad \int_0^{\sqrt{A-1}} d\delta \;=\; \sqrt{A-1} \;\simeq\; \sqrt{A}$

$$q \;\simeq\; \pi B_v \Delta\nu\left(\frac{5}{3}+1\right)\sqrt{A} \qquad \boxed{q \;\simeq\; \frac{8\pi}{3}B_\nu\Delta\nu\sqrt{Lk_\nu(0)}} \quad \text{(Black Center)}$$

For comparison, if the line were all of it "thin", we would have,

$$q_\nu \;=\; 2\pi B_\nu \int_0^1 \underbrace{\left(1-e^{-\frac{s}{u}}\right)}_{=s \text{ everywhere}} udu \;\simeq\; 2\pi B_\nu s$$

Integrating over line, $\quad q \;=\; \int_0^\infty q_\nu d\nu \;=\; \frac{\Delta\nu}{2}\int_{-\infty}^\infty q_\nu d\delta \;=\; \frac{\Delta\nu}{2}2\pi B_\nu \int_{-\infty}^\infty s d\delta$

And since $s = \frac{A}{1+\delta^2}$, $\int_{-\infty}^\infty s d\delta = \pi A$ $\qquad \boxed{q = \pi^2 Lk_{\nu_c}\Delta\nu B_\nu}$ (thin line).

In terms of density and the total absorption cross-section $Q_{TOT} = \int\limits_{line} \alpha_{nm} dv$, we have

$$k_\nu \;=\; N_n Q_\nu \;=\; N_n Q_{\text{TOT}} \frac{2}{\pi \Delta\nu} \frac{1}{1 + \left(2 \frac{\nu - \nu_c}{\Delta\nu/2}\right)^2}$$

$$\begin{aligned} k_\nu(0) \;&=\; \frac{2}{\pi\Delta\nu} N_n Q_{\text{TOT}} \\ \text{so, } Lk_\nu(0) \;&=\; \frac{2}{\pi\Delta\nu} L N_n Q_{\text{TOT}} \\ \text{so, } \boxed{(q)_{\text{Thin}} \;=\; 2\pi B_\nu L N_n Q_{\text{TOT}}} \end{aligned}$$

If this were a black body, we would have in $\Delta\nu$ a radiation,

$$(q_\nu)_{\text{BB}} \;=\; 2\pi B_\nu \int_0^1 (1 - e^{-\infty}) u du \;=\; \pi B_\nu \;\; ; \;\; \frac{(q_\nu)_{\text{Thin}}}{(q_\nu)_{\text{BB}}} \;=\; 2s. \;\; At\; \nu \;=\; \nu_c, s \;=\; A \;=\; k_{\nu_c} L$$

$$\text{So} \left(\frac{q_{\nu_{\text{thin}}}}{q_{\nu_{\text{BB}}}}\right) \;=\; 2Lk_{\nu_c} \;\ll\; 1$$

We can also define the "escape factor", or fraction of light emitted which does escape. For the "blackened-center" line,

$$\beta \;=\; \frac{\frac{8\pi}{3} B_\nu \Delta\nu \sqrt{Lk_{\nu(0)}}}{\pi^2 L k_{\nu_0} \Delta\nu B_\nu} \;=\; \frac{8}{3\pi} \frac{1}{\sqrt{Lk_{\nu_c}}} \ll 1$$

<u>Net</u> radiant heat flux (to the right) in slab of thickness L, at a distance x from the left boundary.

$$q \;=\; q_R - q_L \qquad\qquad \begin{aligned} q_R \;&=\; \frac{8\pi}{3} B_\nu \Delta\nu \sqrt{x k_{\nu_0}} \\ q_L \;&=\; \frac{8\pi}{3} B_\nu \Delta\nu \sqrt{(L-x) k_{\nu_0}} \end{aligned}$$

$$\boxed{q(x) \;=\; \frac{8\pi}{3} B_\nu \Delta\nu \sqrt{k_{\nu_0}} (\sqrt{x} - \sqrt{L-x})}$$

Radiant heat loss per unit volume: $\boxed{\frac{dq}{dx} \;=\; \frac{4\pi}{3} B_\nu \Delta\nu \sqrt{k_{\nu_0}} \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{L-x}}\right)}$

For an almost "transparent" medium:

$$\left.\begin{aligned} q_R(x) \;&=\; \pi^2 x k_{\nu_0} \Delta\nu B_\nu \\ q_L(x) \;&=\; \pi^2 (1-x) k_{\nu_0} \Delta\nu B_\nu \end{aligned}\right\} \boxed{q(x) \;=\; \pi^2 (2x - l) k_{\nu_0} \Delta\nu B_\nu \quad ; \quad \frac{dq}{dx} \;=\; 2\pi^2 k_{\nu_0} \Delta\nu B_\nu}$$

Local Escape Factor:

$$\beta(x) \;=\; \frac{\left(\frac{dq}{dx}\right)}{\left(\frac{dq}{dx}\right)_{\text{Thin}}} \;=\; \frac{2}{3\pi} \left(\frac{1}{\sqrt{k_{\nu_0} x}} + \frac{1}{\sqrt{k_{\nu_0}(L-x)}}\right) \text{ (Must be limited to <1 near } x = 0 \text{ and } x = L)$$

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