

Plasma-Wall Interaction: Sheath and Pre-sheath

Under most conditions, a very thin negative sheath appears in the vicinity of walls, due to accumulation of electrons on the wall. This is in turn necessitated by the need to reduce the electron flux well below the un-inhibited thermal flux $n_e \bar{c}_e/4$, which would lead to very high negative current densities toward the wall. In the limit of an insulating wall, we must have $\Gamma_e = \Gamma_i \ll n_e \bar{c}_e/4$, and the flux equality imposes a specific sheath potential drop (see later). But even where there is net negative current (electron capture) its magnitude is still low enough compared to the thermal value that a negative sheath forms (somewhat against intuition). Any ion arriving at the sheath's edge will be accelerated by the sheath and captured by the wall. This non-zero ion flux requires, by continuity, a means of accelerating ions from conditions far from the wall, unless these conditions already imply a strong wall-directed ion flux. Thus, there is normally a Pre-sheath where the plasma is still neutral, but the potential is already falling toward the wall. The specifics of the pre-sheath vary depending on the situation, and in most cases conditions are 2D or 3D. In contrast, the sheath is almost always so thin as to be 1D, and its physics is simpler and universal, largely independent of outside conditions. For this reason we analyze first the sheath, and it will turn out that its entry condition provides the required wall boundary condition to use for the much larger scale of the pre-sheath.

The Sheath

Because the sheath thickness (a few Debye lengths) is usually much less than any collisional mean free path, we ignore collisions in it. With no magnetic field (or with \vec{B} not nearly parallel to the wall), the electron force balance is between pressure and potential gradients, leading to Boltzmann equilibrium:

$$n_e = n_{es} e^{\frac{e(\phi - \phi_s)}{kT_e}} \quad (1)$$

Where \textcircled{S} is the sheath's edge. Actually, this balance is also valid in the pre-sheath, although less rigorously.

For ions with no scattering or ionizing effects:

$$n_i u_i = n_{is} u_{is} \quad (2)$$

$$m_i \frac{u_i^2}{2} + e\phi = m_i \frac{u_{is}^2}{2} + e\phi_s \quad (3)$$

and eliminating u_i between (2) and (3):

$$n_i = \frac{n_{is}}{\sqrt{1 + \frac{2e(\phi_s - \phi)}{m_i u_{is}^2}}} \quad (4)$$

The net charge density $\rho_{ch} = e(n_i - n_e)$ can then be used in Poisson's equation

$$\frac{d^2 \phi}{dx^2} = \frac{-\rho_{ch}}{\epsilon_0} :$$

$$\frac{d^2\phi}{dx^2} = \frac{e}{\epsilon_0} \left[n_{es} e^{-\frac{e(\phi-\phi_s)}{kT_e}} - \frac{n_{is}}{\sqrt{1 + \frac{2e(\phi_s-\phi)}{m_i u_{is}^2}}} \right] \quad (5)$$

and, actually, $n_{es} = n_{is}$ since this must connect with outside, neutral zone. We can simplify notation by defining non-dimensional variables,

$$\varphi = \frac{e(\phi_s - \phi)}{kT_e} \geq 0 \quad ; \quad \xi = \frac{x}{\lambda_{D_s}} \quad ; \quad \lambda_{D_s} = \sqrt{\frac{\epsilon_0 kT_e}{e^2 n_{es}}} \quad ; \quad M_{is} = \frac{u_{is}}{\sqrt{\frac{kT_e}{m_i}}} \quad (6)$$

This yields,

$$\frac{d^2\varphi}{d\xi^2} = -e^{-\varphi} + \frac{1}{\sqrt{1 + \frac{2\varphi}{M_{is}^2}}} \quad (7)$$

A first integral of (7) is obtained by multiplication times $\frac{d\varphi}{d\xi}$ and integration of the RHS with respect to φ :

$$\frac{1}{2} \left(\frac{d\varphi}{d\xi} \right)^2 = +e^{-\varphi} + \frac{M_{is}^2}{2} \sqrt{1 + \frac{2\varphi}{M_{is}^2}} - C \quad (8)$$

The constant of integration C is obtained by imposing zero slope at s (but only on the magnified sheath scale!). At s we also have $\varphi = 0$, so

$$C = 1 + M_{is}^2 \quad (9)$$

When taking the square root in (8), the $(-)$ sign is appropriate since $\frac{d\phi}{d\xi} < 0$ in the sheath:

$$\frac{d\varphi}{d\xi} = -\sqrt{2} \sqrt{M_{is}^2 \left(\sqrt{1 + \frac{2\varphi}{M_{is}^2}} - 1 \right) - (1 - e^{-\varphi})} \quad (10)$$

Unfortunately, a second integration is not analytically feasible, but important information can be obtained by examining the radicand of (10) near the sheath's edge, i.e. when $\varphi \ll 1$:

$$\begin{aligned} M_{is}^2 \left(\sqrt{1 + \frac{2\varphi}{M_{is}^2}} - 1 \right) - (1 - e^{-\varphi}) &= M_{is}^2 \left[\frac{\varphi}{M_{is}^2} - \frac{1}{8} \left(\frac{2\varphi}{M_{is}^2} \right)^2 + \frac{1}{16} \left(\frac{2\varphi}{M_{is}^2} \right)^3 \cdots \right] \\ - \left(\varphi - \frac{\varphi^2}{2} + \frac{\varphi^3}{6} \cdots \right) &= \frac{1}{2} \left(1 - \frac{1}{M_{is}^2} \right) \varphi^2 + \left(\frac{1}{2M_{is}^4} - \frac{1}{6} \right) \varphi^3 + \cdots \end{aligned} \quad (11)$$

The leading term (in φ^2) must be positive or zero for real solutions near the sheath edge:

$$\boxed{M_{is}^2 \geq 1} \quad (12)$$

and this is called the Bohm sheath criterion. Supersonic ion entry is uncommon, so we will accept the weaker criterion.

$$\boxed{M_{is} = 1} \quad (13)$$

When this is satisfied, the next term in (11) is $\frac{1}{3}\varphi^3 > 0$, and the sheath potential profile is real. Thus, the ions enter the sheath at their (isothermal) speed of sound,

$$\boxed{u_{is} = \sqrt{\frac{kT_e}{m_i}}} \quad (14)$$

and this is the boundary condition needed for the outer (pre-sheath) solution. Once φ exceeds unity ($\phi_s - \phi \geq kT_e$), the expression (10) simplifies (roughly) to

$$\frac{d\varphi}{d\xi} = -\sqrt{2}\sqrt{M_{is}^2 \frac{2\varphi}{M_{is}}} = 2^{3/4}\varphi^{1/4} \quad (15)$$

$$\uparrow M_{is} = 1$$

which can be integrated:

$$\begin{aligned} -\varphi^{-1/4}d\varphi &= 2^{3/4}d\xi \\ -\frac{4}{3}\varphi^{3/4} &= 2^{3/4}\xi + D \end{aligned}$$

where, using $\varphi = 0$ at s , $D = -2^{3/4}\xi_s$

$$\xi_s - \xi = \frac{2^{5/4}}{3}\varphi^{3/4} \quad (16)$$

In particular, at the wall $\xi = \xi_w$, $\varphi = \varphi_w$, so the sheath thickness is $\xi_s - \xi_w = \frac{2^{5/4}}{3}\varphi_w^{3/4}$. In physical variables,

$$\frac{x_s - x_w}{\lambda_{D_s}} = \frac{2^{5/4}}{3} \left(\frac{e\Delta\phi_{sh}}{kT_e} \right)^{3/4} \quad (17)$$

so the sheath increases in thickness with the $3/4$ power of the sheath potential drop. Putting also in (17) the definition of λ_{D_s} , we can reorganize it in the form

$$j_i = en_{es}u_{is} = \frac{4\sqrt{2}}{9} \sqrt{\frac{e}{m_i}} \epsilon_0 \frac{\Delta\phi_{sh}^{3/2}}{(x_s - x_w)^2} \quad (18)$$

which is the Child-Langmuir equation for space-charge limited current.

The Collisional Pre-sheath

An alternative derivation of the Bohm criterion could be arrived at in the following way. Start with steady 1D flow and impose continuity of ion flow and quasineutrality,

$$\frac{d}{dx}(n_e u_i) = 0 \quad \text{and} \quad n_e = n_i$$

And include momentum conservation for both ions and electrons,

$$\begin{aligned} n_e m_i u_i \frac{du_i}{dx} + \frac{dp_i}{dx} &= n_e e E_x - m_i n_e \nu_{in} u_i \\ \cancel{n_e m_e u_e} \frac{du_e}{dx} + \frac{dp_e}{dx} &= -n_e e E_x - m_e n_e \nu_{en} u_e \end{aligned}$$

Adding these equations and combining with continuity, we can write,

$$\frac{d}{dx} \left(m_i n_e u_i^2 + \underbrace{p_i + p_e}_{n_e k(T_e + T_i)} \right) = 0 - m_i n_e \nu_{in} u_i - \cancel{m_e n_e \nu_{en} u_e}$$

or, with $T_e \gg T_i$,

$$(\cancel{n_e u_i}) \frac{d}{dx} \left(m_i u_i + \frac{kT_e}{u_i} \right) \simeq -m_i \nu_{in} (\cancel{n_e u_i})$$

So, we get the same sheath entry velocity

$$\boxed{u_{is} = \sqrt{\frac{kT_e}{m_i}} = u_{\text{Bohm}}}$$

Particle and energy flux to a wall

(A) Random particle flux across a plane

Assume a Maxwellian distribution:

$$f = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{\overbrace{m(w_x^2 + w_y^2 + w_z^2)}^{w^2}}{2kT}} \quad (19)$$

The forward directed flux across a plane (say, the yz plane) is,

$$\Gamma_x = \int_{w_y=-\infty}^{\infty} dw_y \int_{w_z=-\infty}^{\infty} dw_z \int_0^{\infty} w_x f(\vec{w}) dw_x dw_y dw_z \quad (20)$$

It is actually better to use spherical velocity conditions, where $w_x = w \cos \theta$ and $dw_x dw_y dw_z = d^3 w = 2\pi w^2 \sin \theta dw d\theta$:

$$\Gamma_x = \int_{w=0}^{\infty} \int_{\theta=0}^{\pi/2} f(w) 2\pi w^3 \sin \theta \cos \theta d\theta dw \quad (21)$$

$$\text{Integrate first on } \theta : \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \int_0^{\pi/2} d\left(\frac{\sin^2 \theta}{2}\right) = \frac{1}{2} \quad (22)$$

$$\Gamma_x = \pi \int_{w=0}^{\infty} f(w) w^3 dw = \pi n \left(\frac{m}{2\pi kT} \right)^{3/2} \int_0^{\infty} e^{-\frac{mw^2}{2kT}} w^3 dw$$

Change variables: $\frac{mw^2}{2kT} = \zeta$, $w dw = \frac{kT}{m} d\zeta$

$$\Gamma_x = \pi n \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{2kT}{m} \frac{kT}{m} \underbrace{\int_0^{\infty} \zeta e^{-\zeta} d\zeta}_1 = \frac{1}{\sqrt{2\pi}} n \sqrt{\frac{kT}{m}}$$

$$\Gamma_x = n\sqrt{\frac{kT}{2\pi m}} \quad (23)$$

This is usually written in terms of the mean value of the velocity $\bar{c} \equiv \frac{1}{n} \int_0^\infty \int_0^\pi wf(w)2\pi w^2 \sin\theta d\theta dw$, which, with a similar calculation gives,

$$\boxed{\bar{c} = \sqrt{\frac{8kT}{\pi m}}} \quad (24)$$

By division of (5) and (6), we obtain

$$\boxed{\Gamma_x = \frac{n\bar{c}}{4}} \quad (\text{note } x \text{ is an arbitrary direction in this case}) \quad (25)$$

(B) Random flux of kinetic energy across a plane

We now want to calculate:

$$\Gamma_E = \int_{w=0}^\infty \int_{\theta=0}^{\pi/2} \left(\frac{1}{2}mw^2\right)f(w)(w\cos\theta)2\pi w^2 \sin\theta d\theta dw \quad (26)$$

$$= m\frac{\pi}{2} \int_{w=0}^\infty w^5 f(w)dw = m\frac{\pi}{2} n \left(\frac{m}{2\pi kT}\right)^{3/2} \int_{w=0}^\infty w^5 e^{-\frac{mw^2}{2kT}} dw \quad (27)$$

Using the name change of the variable of before,

$$\Gamma_E = m\frac{\pi}{2} n \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{1}{2} \left(\frac{2kT}{m}\right)^3 \underbrace{\int_0^\infty \zeta^2 e^{-\zeta} d\zeta}_{-\zeta^2 e^{-\zeta} \Big|_0^\infty + 2 \int_0^\infty \zeta e^{-\zeta} d\zeta = 2}$$

$$\Gamma_E = 2kT\sqrt{\frac{kT}{2\pi m}} \quad (28)$$

$$\text{and comparing to (6)} \quad , \quad \boxed{\Gamma_E = \Gamma_x 2kT} \quad (29)$$

So the average particle that crosses the plane carries an energy $2kT$. This is more than the average energy per particle ($\frac{3}{2}kT$) because faster particles cross more often.

(C) Particle flux to a repelling wall

This could be electrons approaching a negatively charged wall, with a sheath in front. We do the calculation outside the sheath, and, if there are no collisions, the same flux will cross any plane, including the wall plane. The calculation is like in part (A), but now we notice that not all particles reach the wall: they must have enough x -directed energy to overcome the repulsion. If $q\phi_w > 0$ (for example, electrons, with $q = -e$ to a negative wall), we need,

$$\frac{m(w\cos\theta)^2}{2} \geq q\phi_w \quad (30)$$

and for a fixed w ,

$$\theta_{MAX} = \cos^{-1} \sqrt{\frac{2q\phi_w}{mw^2}} \quad (31)$$

and so the inner integration (on θ) now gives,

$$\int_0^{\theta_{MAX}} \sin\theta \cos\theta d\theta = \int_0^{\theta_{MAX}} d\left(\frac{-\cos^2\theta}{2}\right) = \frac{1}{2} \left(1 - \cos^2\theta_{MAX}\right) = \frac{1}{2} \left(1 - \frac{2q\phi_w}{mw^2}\right) \quad (32)$$

and for the integration (on w), $w_{MIN} = \sqrt{\frac{2q\phi_w}{m}}$, from (14)

$$\begin{aligned} \Gamma &= \int_{\sqrt{\frac{2q\phi_w}{m}}}^{\infty} f(w) w^3 \left(1 - \frac{2q\phi_w}{mw^2}\right) dw \\ &= \pi n \left(\frac{m}{2\pi kT}\right)^{3/2} \int_{\sqrt{\frac{2q\phi_w}{m}}}^{\infty} e^{-\frac{mw^2}{2kT}} w^3 \left(1 - \frac{2q\phi_w}{mw^2}\right) dw \end{aligned} \quad (33)$$

Using again $\zeta = \frac{mw^2}{2kT}$:

$$\Gamma_x = \underbrace{\pi n \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{1}{2} \left(\frac{2kT}{m}\right)^2}_{\frac{n\bar{c}}{4}} \int_{\frac{q\phi_w}{kT}}^{\infty} e^{-\zeta} \zeta \left(1 - \frac{q\phi_w}{kT} \frac{1}{\zeta}\right) d\zeta$$

For the last integral, integrate by parts ($e^{-\zeta} d\zeta = -d(e^{-\zeta})$),

$$\begin{aligned} \Gamma &= \frac{n\bar{c}}{4} \left[\underbrace{-\left(\zeta - \frac{q\phi_w}{kT}\right) e^{-\zeta}}_0 \Big|_{\frac{q\phi_w}{kT}}^{\infty} + \underbrace{\int_{\frac{q\phi_w}{kT}}^{\infty} e^{-\zeta} d\zeta}_{e^{-\frac{q\phi_w}{kT}}} \right] \\ \therefore \quad \Gamma &= \frac{n\bar{c}}{4} e^{-\frac{q\phi_w}{kT}} \leq \frac{n\bar{c}}{4} \end{aligned} \quad (34)$$

Clearly, the wall repulsion reduces the flux (strongly if $q\phi_w > kT$).

(D) Energy flux to a repelling wall

The same considerations apply now. We start with (8), but with the limits modified as in part C. The algebra is straightforward, although tedious, and the result is:

$$\Gamma_E = \underbrace{\frac{n\bar{c}}{4} e^{-\frac{q\phi_w}{kT}}}_{\Gamma} (2kT + q\phi_w) \quad (35)$$

So now the average crossing particle carries an energy $2kT + q\phi_w$. The extra $q\phi_w$ per particle is what is required for it to actually reach the wall, while the $2kT$ part is what it would carry in thermal form with no repulsion.

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