The Electron Energy Equation

Since $\vec{v}_e$ is often very different from other species mean velocities, it makes sense to formulate the energy equation in terms of T'_e, defined as $n_e \frac{3}{2} k T'_e = \left\langle \frac{1}{2} m_e c_e^2 \right\rangle_e$, $\vec{c}_e = \vec{w} - \vec{v}_e$. We found before (with no inelastic effects)

$$\frac{\partial}{\partial t}\left(n_e \frac{3}{2} k T'_e\right) + \nabla \cdot \left(n_e \vec{v}_e \frac{3}{2} k T'_e + \vec{q}'_e\right) + \underset{P}{\Rightarrow} : \nabla \vec{v}_e \;=\; \sum_r E'_{re} \tag{1}$$

and, at least for Maxwellian collision (but generalizable to others),

$$E'_{re} \;=\; n_e v_{er} \mu_{re} \left[\frac{3k(T'_r - T'_e)}{m_e + m_r} + \frac{m_r}{m_r + m_e}(\vec{v}_r - \vec{v}_e)^2\right] \tag{2a}$$

$$E'_{re} \;\cong\; n_e v_{er} m_e \left[\frac{3k(T'_r - T'_e)}{m_r} + (\vec{v}_r - \vec{v}_e)^2\right] \tag{2b}$$

where (2b) results from $m_e << m_r$.

One first observation is that the "heat transfer" portion of this, namely

$$El \;=\; n_e v_{er} \frac{2m_e}{m_r} \frac{3}{2} k (T'_r - T'_e) \tag{3}$$

can be thought of as transferring the mean thermal energy difference per particle, $\frac{3}{2}k(T'_r - T'_e)$, per collision, but with a very poor efficiency

$$n_{el} \;=\; \frac{2m_e}{m_r} << 1 \tag{4}$$

In other words, while Maxwellian distributions about T'_e and T'_r established in a few e.g. e-e and r-r collisions, it takes about $\frac{m_r}{2m_e}$ collisions (tens to hundreds of thousands) to drive T'_e

toward T'_r. In practice, a good approximation is that there are separate Maxwellian populations for electrons (at T'_e) vs. the heavy species (at close to the same T'_r, since the collisional efficiency among them $\frac{2\mu_{rs}}{m_r+m_s}$ is the of order 1).

We next examine the irreversible second term in (2b). The summation over r-species includes ions and neutrals, and we assume a single kind of each. We then have a dissipation

$$D = n_e m_e [v_{ei}(\vec{v}_i - \vec{v}_e)^2 + v_{en}(\vec{v}_n - \vec{v}_e)^2] \tag{5}$$

The electron and ion current densities are

$$\vec{j}_e = -en_e\vec{v}_e \tag{6a}$$

$$\vec{j}_i = en_e\vec{v}_1 \tag{6b}$$

$$\text{and so} \quad D = n_e m_e \left[v_{ei}\left(\frac{\vec{j}}{en_e}\right)^2 + v_{en}\left(\vec{v}_n + \frac{\vec{j}}{en_e}\right)^2 \right] \tag{7}$$

This expression simplifies several limits:

(a) $v_n << v_i, v_e$

$$D \cong n_e m_e \left[v_{ei}\frac{j^2}{e^2 n_e^2} + v_{en}\frac{j_e^2}{e^2 n_e^2} \right] = \frac{j^2}{\left(\frac{e^2 n_e}{m_e v_{ei}}\right)} + \frac{j_e^2}{\left(\frac{e^2 n_e}{m_e v_{en}}\right)}$$

The quantities in the denominator are the conductivities if only ei or en collisions occurred:

$$D \cong \frac{j^2}{\sigma_{ei}} + \frac{j_e^2}{\sigma_{en}} \tag{8}$$

and, in particular

(a.1) For a neutral-dominated gas ($v_{en} >> v_{ei}$) (Hall Thruster), $D \cong \frac{j_e^2}{\sigma}$ (9)

(a.2) For a Coulomb-dominated gas ($v_{ei} >> v_{en}$)$D \cong \frac{j^2}{\sigma}$ (10), (a.3) If $j_i << j_e, j_e \cong j$ and $D \cong \frac{j^2}{\sigma}$, with $\sigma = \frac{e^2 n_e}{m_e(v_{en}+v_{ei})}$).

(b) When density is relatively high, ions and neutrals couple strongly and $\vec{v}_n \cong \vec{v}_i$ (as in MPD thrusters or MHD generators). In that case (5) yields

$$D \cong n_e m_e (v_{ei} + v_{en})(\vec{v}_i - \vec{v}_e)^2 \;=\; \frac{j^2}{\sigma} \tag{11}$$

with

$$\sigma \;=\; \frac{e^2 n_e}{m_e(v_{ei} + v_{en})} \tag{12}$$

One approximation which is routinely made is to neglect the <u>viscous dissipation</u> of the electron gas, i.e., the contribution of the off-diagonal terms in $\underset{P}{\Rightarrow}: \nabla \vec{v}_e$:

$$\underset{P'_e}{\Rightarrow} \cdot \nabla \vec{v}_e \cong P'_e \nabla . \vec{v}_e \tag{13}$$

where $P'_e \;=\; n_e k T'_e$ is the scalar pressure (the trace of $\underset{P'_e}{\Rightarrow}$). Breaking this into $\nabla \cdot (n_e k T'_e \vec{v}_e) - \vec{v}_e \cdot \nabla P'_e$ and substituting in (1), we get

$$\frac{\partial}{\partial t}\left(n_e \frac{3}{2} k T'_e\right) + \nabla \cdot \left(n_e v_e \frac{5}{2} k T'_e + \vec{q}'_e\right) \;=\; D + El + \vec{v}_e \cdot P'_e \tag{14}$$

with D given by (7) and El given by (3).

Finally, although we will not prove it here, it stands to reason that the heat flux vector $\vec{q}'_e = n_e \left\langle \frac{1}{2} m_e c_e^2 \vec{c}_e \right\rangle_e$ will be expressible in the form of a Fourier law

$$\vec{q}'_e \;=\; -K_e(T_e) \nabla T'_e \tag{15}$$

where K_e is the electron thermal conductivity. Note that a simple form like this may not be accurate when the alternative definition T_e of temperature is used.

16.55 Ionized Gases
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